4.1 INTRODUCTION TO SYSTEMS OF ODE's

PREVIOUSLY given some relationship between a function $y(t)$ and its derivatives, solve for y.

First-order: $y'(t) = f(t, y(t))$

How to visualize higher-order eqs?

TODAY several interrelated ODEs, solve for several functions

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -x'(t) \end{cases} \qquad \begin{cases} x'(t) = x(t) + 2y(t) \\ y'(t) = 3x(t) + 4y(t) \end{cases}$$

FACT: any ~~eq~~ diff. eq. can be recast as a 1st order system ("you can trade order for dimension")

EXAMPLE: transform the diff. eq. into an equivalent 1st-order system

$$x'' + 3x' + 7x = t^2$$

Let $x_1(t) := x(t)$ and $x_2(t) := x'(t)$

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = t^2 - 7x_1(t) - 3x_2(t) \end{cases}$$

$$x_2' = x'' = t^2 - 7x - 3x'$$
$$= t^2 - 7x_1 - 3x_2$$

EXAMPLE: $x^{(3)} - 2x'' + x' = 1 + te^t$ $\qquad x(t)$

$$x^{(3)} = 2x'' - x' + 1 + te^t$$

$$x_2' = 2x_3 - x_2 + 1 + te^t$$

Let
$$x_1 = x$$
$$x_2 = x'$$
$$x_3 = x''$$

then
$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 2x_3 - x_2 + 1 + te^t \end{cases}$$

$$\frac{d}{dt}(x_1, x_2, x_3) = F(x_1, x_2, x_3)$$

THREE EXAMPLES

1) HARMONIC OSCILLATOR

$$m x'' + c x' + K x = 0$$

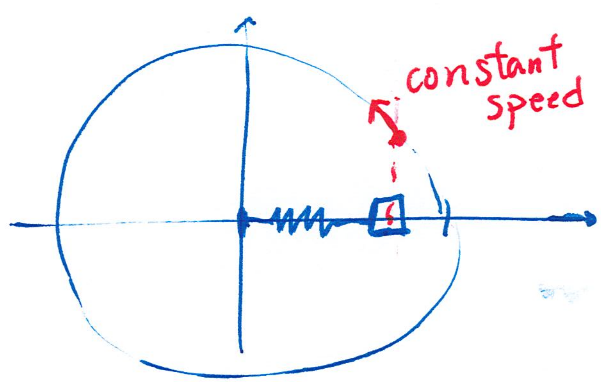
$$\begin{aligned} x_1 &= x \\ x_2 &= x' \end{aligned} \implies \begin{cases} x_1' = x_2 \\ x_2' = \dfrac{-c x_2 - K x_1}{m} \end{cases}$$

(keep track of both position and velocity)

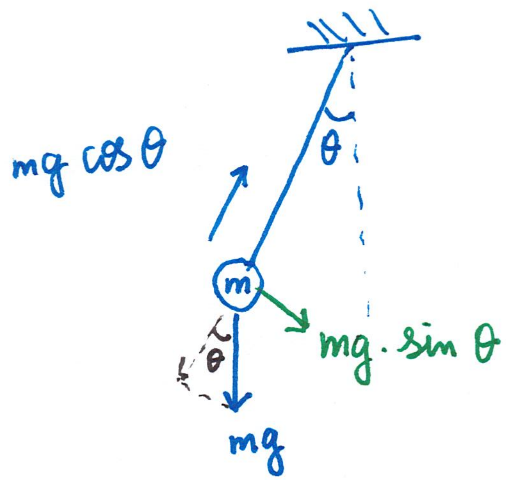
For undamped motion, say

$$x'' + x = 0,$$

the movement is a projection of a uniform circular movement. The plane (x_1, x_2) is called PHASE SPACE.



2) PENDULUM

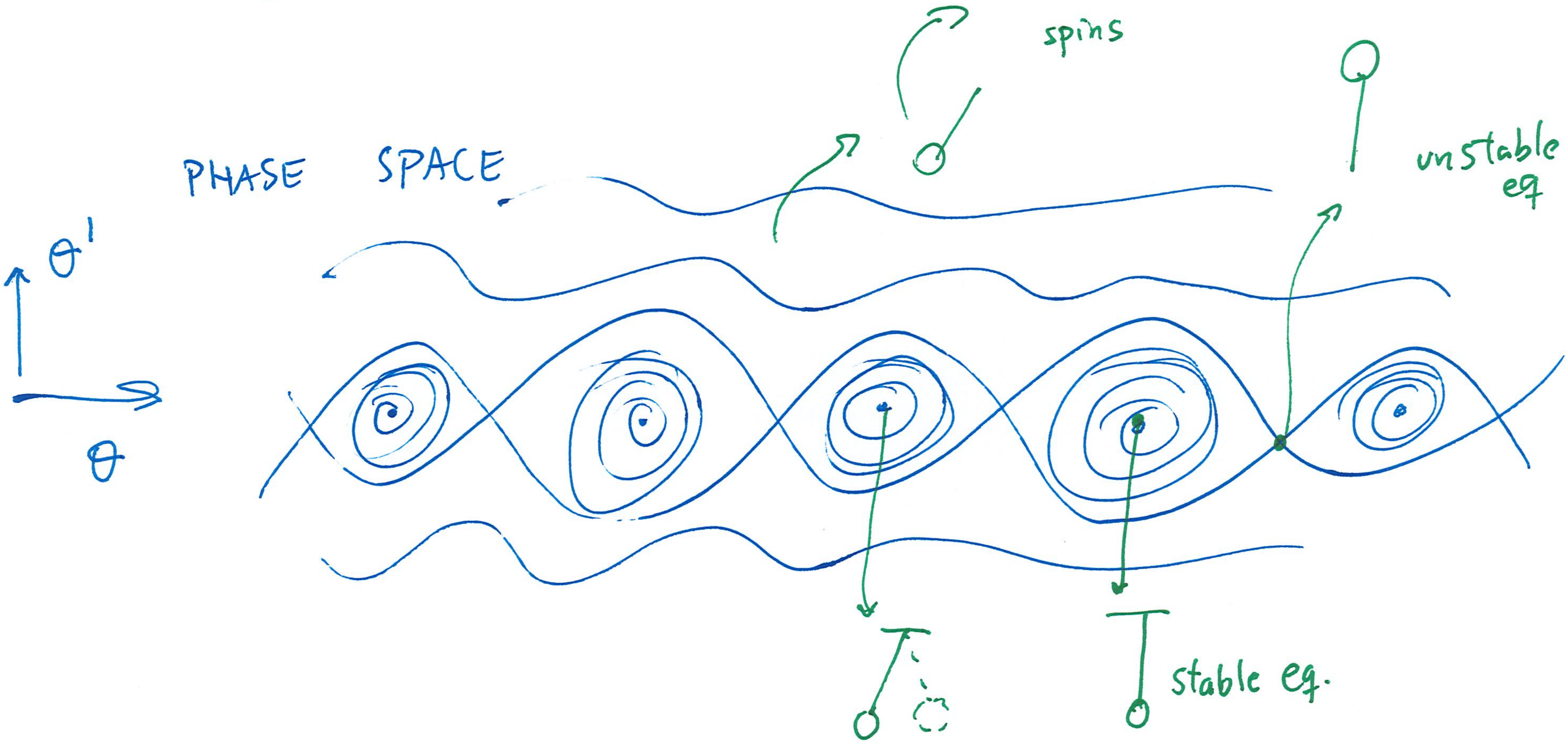


$$\theta'' = -mg \sin\theta$$

(nonlinear)
(but looks like $\theta'' = -\theta$ for small θ)

$$\begin{cases} \theta_1' = \theta_2 \\ \theta_2' = -mg \sin\theta_1 \end{cases}$$

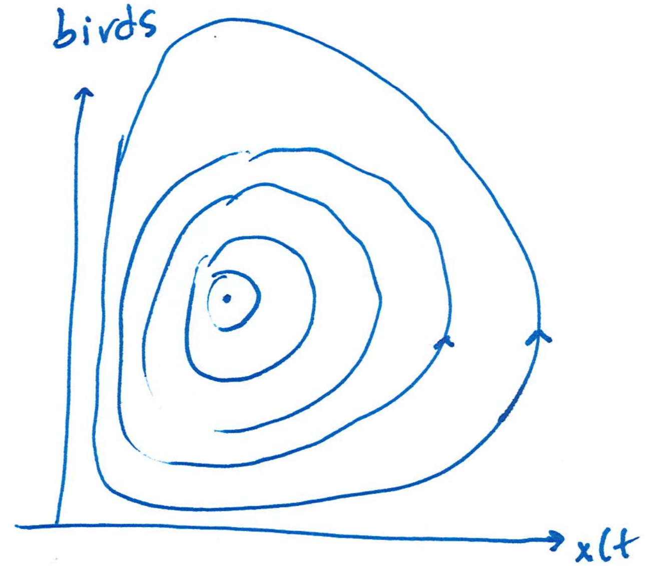
PHASE SPACE



3) PREDATOR-PREY MODEL (A.K.A. LOTKA-VOLTERRA MODEL)

$k, \ell, a, b > 0$

$$\begin{cases} x'(t) = kx(t) - a\,x(t)y(t) \\ y'(t) = -\ell y(t) + b\,x(t)y(t) \end{cases}$$



$x(t)$ = population of prey (insects)

$y(t)$ = pop of predator (birds)

EXAMPLE: Solve for $x(t)$ and $y(t)$ in

$$\begin{cases} x' = -2y & \text{A} \\ y' = \frac{1}{2}x & \text{B} \end{cases}$$

and sketch the solution curves.

Since the coefficients are constant, can replace the system by a 2nd order eq.

Solve for y in A, plug into B

$$x' = -2y \implies y = -\frac{1}{2}x'$$

$$y' = \frac{1}{2}x \implies -\frac{1}{2}x'' = \frac{1}{2}x \implies x'' + x = 0$$

$$\implies x(t) = A\cos t + B\sin t$$

$$y(t) = -\frac{1}{2}x'(t) = \cancel{\frac{B}{2}\cos t}$$

$$= -\frac{B}{2}\cos t + \frac{A}{2}\,\cancel{\cos}\sin t$$

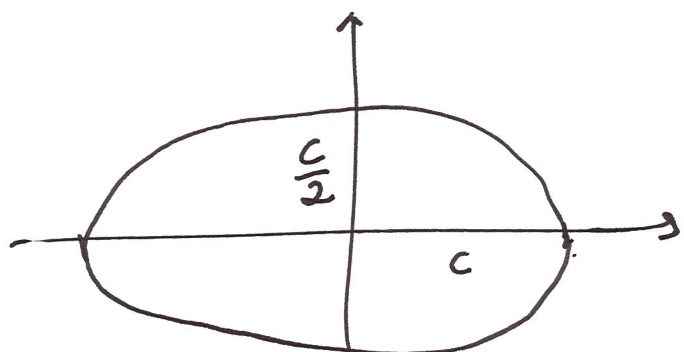
Not clear how to sketch the curves.

Rewrite $x(t) = C \cdot \cos(t-\alpha)$

$$y(t) = -\frac{1}{2}x'(t) = \frac{C}{2}\sin(t-\alpha)$$

Notice that $\dfrac{x^2}{C^2} + \dfrac{y^2}{\left(\frac{C}{2}\right)^2} = \cos^2(t-\alpha) + \sin^2(t-\alpha)$
$= 1,$

So $(x(t), y(t))$ is in the ellipse



So the phase plane looks like this