4.2 THE METHOD OF ELIMINATION

EXAMPLE: find the general solution of the system

$$\begin{cases} x' = x - 2y & (A) \\ y' = 2x - 3y & (B) \end{cases}$$

Solve for x in eq. (B), plug into (A).

$$x = \frac{1}{2}(y' + 3y) \quad (\star)$$

$$\frac{1}{2}(y'+3y)' = \frac{1}{2}(y'+3y) - 2y$$

$$y'' + 3y' = y' + 3y - 4y$$

$$y'' + 2y' + y = 0$$

Characteristic eq. $r^2 + 2r + 1 = 0 \iff (r+1)^2 = 0.$

General solution $y(t) = (A + Bt)e^{-t}.$

Plug into $(\star)$ to find x:

$$x = \frac{1}{2}(y' + 3y) = \frac{1}{2}(-A - Bt + B + 3A + 3Bt)e^{-t}$$

$$\boxed{\begin{aligned} x &= \left(A + \frac{B}{2} + Bt\right)e^{-t} \\ y &= (A + Bt)e^{-t} \end{aligned}}$$

Solve for y in eq. (A), plug into (B).

$$y = \frac{1}{2}(x - x') \quad (\star\star)$$

$$\frac{1}{2}(x - x')' = 2x - \frac{3}{2}(x - x')$$

$$x'' + 2x' + x = 0$$

$$x = (A + Bt)e^{-t}$$

Plug into $(\star\star)$ to find y

$$y = \frac{1}{2}e^{-t}(A + Bt + A + Bt - B)$$

$$y = \left(A - \frac{B}{2} + Bt\right)e^{-t}$$

$$\boxed{\begin{aligned} x &= (A + Bt)e^{-t} \\ y &= \left(A - \frac{B}{2} + Bt\right)e^{-t} \end{aligned}}$$

Can you see that both answers are the same?

EXAMPLE: solve the IVP

$$x' = x - 2y, \quad y' = 2x - 3y, \quad x(0) = 0, \quad y(0) = 1$$

$$\boxed{\begin{aligned} x &= \left(A + \frac{B}{2} + Bt\right)e^{-t} \\ y &= (A + Bt)e^{-t} \end{aligned}} \quad \left\{ \begin{aligned} 0 &= A + \frac{B}{2} \\ 1 &= A \end{aligned} \right\} \Rightarrow B = -2 \qquad x = -2te^{-t}, \; y = (1-2t)e^{-t}$$

$$\boxed{\begin{aligned} x &= (A + Bt)e^{-t} \\ y &= \left(A - \frac{B}{2} + Bt\right)e^{-t} \end{aligned}} \quad \left\{ \begin{aligned} 0 &= A \\ 1 &= A - \frac{B}{2} \end{aligned} \right\} \Rightarrow B = -2 \qquad x = -2te^{-t}, \; y = (1-2t)e^{-t}$$

EXAMPLE: Solve the IVP

$$x'' - 2y' + 3x = 0, \quad y'' + 2x' + 3y = 0, \quad x(0) = 4, \quad y(0) = x'(0) = y'(0) = 0$$

We'll use polynomial operators: denote $Dx = x'$, $D^2x = x''$, $Dy = y'$, $D^2y = y''$.

Rewrite the system as

$$\begin{cases} (D^2+3)x - 2Dy = 0 & (A) \\ 2Dx + (D^2+3)y = 0 & (B) \end{cases}$$

Eliminate one of the variables as if the D's were numbers (but they are not).

Apply $2D$ to (A) and D^2+3 to (B).

$$2D(D^2+3)x - 4D^2y = 0$$

$$(D^2+3)2Dx + (D^2+3)(D^2+3)y = 0$$

$$x(t) = 3\cos t + \cos 3t$$

$$y(t) = 3\sin t - \sin 3t$$

Subtract the 1st from the second:

$$[(D^2+3)(D^2+3) + 4D^2]y = 0$$

$$[D^4 + 10D^2 + 9]y = 0$$

$$(D^2+3)(D^2+1)y = 0$$

$$y(t) = A\cos t + B\sin t + C\cos 3t + D\sin 3t$$

$$y'(t) = B\cos t - A\sin t + 3D\cos 3t - 3C\sin 3t$$

$$y''(t) = -A\cos t - B\sin t - 9C\cos 3t - 9D\sin 3t$$

Plug into $y'' + 2x' + 3y = 0$ and solve for x'.

$$2x' = -y'' - 3y = \cos t\,(A - 3A) + \sin t\,(B - 3B) + \cos 3t\,(9C - 3C) + \sin 3t\,(9D - 3D)$$

$$x' = -A\cos t - B\sin t + 3C\cos 3t + 3D\sin 3t$$

$$x = B\cos t - A\sin t - D\cos 3t + C\sin 3t + E$$

Use the initial conditions to find A, B, C, D, E:

$$\left.\begin{aligned} x(0) = 4 &\Rightarrow B - D + E = 4 \\ x'(0) = 0 &\Rightarrow -A + 3C = 0 \\ y(0) = 0 &\Rightarrow A + C = 0 \\ y'(0) = 0 &\Rightarrow B + 3D = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &A = C = 0 \\ &B = -3D \\ &E = 4(1 + D) \end{aligned}$$

$$x(t) = -3D\cos t - D\cos 3t + E$$

$$y(t) = -3D\sin t + D\sin 3t$$

To find D and E, plug into $x'' - 2y' + 3x = 0$.

$$(3D\cos t + 9D\cos 3t) - 2(-3D\cos t + 3D\cos 3t) + 3(-3D\cos t - D\cos 3t + E) = 0$$

$\Rightarrow E = 0$ (everything cancels)

Using $E = 4(1 + D)$, we get $D = -1$.

$$\boxed{\begin{aligned} x(t) &= 3\cos t + \cos 3t \\ y(t) &= 3\sin t - \sin 3t \end{aligned}}$$

DEGENERATE SYSTEMS

Sometimes a system of differential equations can have no solution at all, sometimes it can have infinitely many linearly independent solutions. If any of these happen, the system is said to be **degenerate**.

EXAMPLES

$$\begin{cases} x' + y' = 1 \\ 2x' + 2y' = 0 \end{cases}$$ degenerate, no solutions

$$\begin{cases} x' + 2x + y' + 2y = e^{-3t} & A \\ x' + 3x + y' + 3y = e^{-2t} & B \end{cases}$$

$B - A$: $x + y = e^{-2t} - e^{-3t}$

$3A - 2B$: $x' + y' = -2e^{-2t} + 3e^{-3t}$ ← this equation is redundant

degenerate, infinitely many LI solutions

EXAMPLE: find the general solution of

$$\begin{cases} x' = x + 2y + z & \text{(A)} \\ y' = 6x - y & \text{(B)} \\ z' = -x - 2y - z & \text{(C)} \end{cases}$$

Solve for x in (B), plug into (A) and (C) to get a system in y and z.

$x = \frac{1}{6}(y' + y)$ (X)

$\frac{1}{6}(y' + y)' = \frac{1}{6}(y' + y) + 2y + z$ (A1)

$z' = -\frac{1}{6}(y' + y) - 2y - z$ (B1)

Solve for z in (A1), plug into (B1)

$z = \frac{1}{6}y'' - \frac{13}{6}y$ (Z)

$\frac{1}{6}y''' - \frac{13}{6}y' = -\frac{1}{6}(y' + y) - 2y - (\frac{1}{6}y'' - \frac{13}{6}y)$

$y''' + y'' - 12y' = 0$

Characteristic equation: $r^3 + r^2 - 12r = 0$

$\Rightarrow r(r^2 + r - 12) = 0 \Rightarrow r(r+4)(r-3) = 0$

$\boxed{y = A + Be^{-4t} + Ce^{3t}}$

Use (X) and (Z) to compute x and z now that we know y.

$y = A + Be^{-4t} + Ce^{3t}$

$y' = -4Be^{-4t} + 3Ce^{3t}$

$y'' = 16Be^{-4t} + 9Ce^{3t}$

(X) $\boxed{x = \frac{A}{6} - \frac{B}{2}e^{-4t} + \frac{2C}{3}e^{3t}}$

(Z) $\boxed{z = -\frac{13A}{6} + \frac{B}{2}e^{-4t} - \frac{2C}{3}e^{3t}}$