5.1 LINEAR INDEPENDENCE OF VECTOR-VALUED FUNCTIONS

RECALL for $y''+p(t)y'+q(t)y=0$ the general solution has the form $y(t)=Ay_1(t)+By_2(t)$, where y_1 and y_2 are LI solutions (not multiples of each other)

For example, $y''+2y'+y=0$ has general solution $y(t)=Ae^{-t}+Bte^{-t}$.

REMARK the function $2t$ is a multiple of the function t, but the function t^2 is not. In other words, $\{t,2t\}$ is LD and $\{t,t^2\}$ is LI.

TODAY general solutions of SYSTEMS of linear equations

EXAMPLE $y''+2y'+y=0$ can be recast as a system. Let $x_1:=y$, $x_2:=y'$

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_2 - x_1 \end{cases}$$

general solution: $x_1(t)=(A+Bt)e^{-t}$

$x_2(t)=(-A+B-Bt)e^{-t}$

In matrix form: let $\mathbf{x}(t):=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, then $\mathbf{x}'(t)=\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\mathbf{x}(t)$.

General solution: $\mathbf{x}(t) = A\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + B\begin{bmatrix} te^{-t} \\ (1-t)e^{-t} \end{bmatrix}$.

Compare $y(t) = Ae^{-t} + Bte^{-t}$ 2nd order equation, two LI solutions

$$\mathbf{x}(t) = A\begin{bmatrix} e^{t} \\ -e^{-t} \end{bmatrix} + B\begin{bmatrix} te^{-t} \\ (1-t)e^{-t} \end{bmatrix}.$$

1st order SYSTEM of two equations, two LI solutions

FACT Let P be an $n\times n$ matrix. Consider the system $\mathbf{x}' = P\mathbf{x}$. Its general solution has the form

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + \cdots + C_n\mathbf{x}_n(t),$$ where $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are LI solutions.

RECALL a set of functions $y_1, \dots, y_n$ is linearly independent if none of them is a linear combination of the others. The same definition applies to vector-valued functions.

LI	LD
$\{e^{t}, e^{2t}, e^{3t}\}$	$\{e^{t}, 2e^{t}, e^{3t}\}$
$\{\cos x, \sin x, \sin 2x\}$	$\{\cos x, \sin x, \cos x - \sin x\}$
$\{e^{t}, te^{t}, t^2e^{t}\}$	$\{te^{t}, te^{t}, t^2e^{t}\}$
$\left\{\begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}, \begin{bmatrix} e^{2t} \\ 3e^{2t} \end{bmatrix}\right\}$	$\left\{\begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}, \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix}\right\}$

HOW DO WE TELL IF A GIVEN SET OF SOLUTIONS IS LI ?

THEOREM: Consider a linear system of differential equations $\mathbf{x}' = P(t)\mathbf{x}$, with n equations.
Let $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ be solutions of $\mathbf{x}' = P(t)\mathbf{x}$. Then $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are LI if and only

$$\Bigg| \underbrace{\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ \cdots\ \mathbf{x}_n(t)}_{\substack{n\times n \text{ matrix with} \\ \text{columns } \mathbf{x}_1, \dots, \mathbf{x}_n}} \Bigg| \neq 0. \qquad \text{(Wronskian determinant } W(t))$$

EXAMPLES

1) $\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x}(t).$ Two solutions: $\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ and $\begin{bmatrix} te^{-t} \\ (1-t)e^{-t} \end{bmatrix}$.

$$\begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix} = e^{-2t} \begin{vmatrix} 1 & t \\ -1 & 1-t \end{vmatrix} = e^{-2t}[1-t-(-t)] = e^{-2t} \neq 0$$

$$= e^{-t} \begin{vmatrix} 1 & te^{-t} \\ -1 & (1-t)e^{-t} \end{vmatrix} = e^{-2t}\big(1-t-(-t)\big)$$

$$= e^{-t}e^{-t} \begin{vmatrix} 1 & t \\ -1 & 1-t \end{vmatrix} \qquad = e^{-2t}$$

2) $\mathbf{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\mathbf{x}$

Check that these are LI solutions:

$$\mathbf{x}_1(t) = e^{t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x}_2(t) = e^{3t}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{x}_3(t) = e^{5t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Let P be the matrix in the equation.

Check that $\mathbf{x}_1(t)$ is a solution:

$$\mathbf{x}_1' \overset{?}{=} P\mathbf{x}_1$$

$$\mathbf{x}_1'(t) = e^{t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$P\mathbf{x}_1(t) = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}e^{t}$$

$$= e^{t}\left\{ 2\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \right\}$$

$$= e^{t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \mathbf{x}_1'(t)$$

Check that $\mathbf{x}_2(t)$ is a solution:

$$\mathbf{x}_2' \overset{?}{=} P\mathbf{x}_2$$

$$\mathbf{x}_2'(t) = 3e^{3t}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = e^{3t}\begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix}$$

$$P\mathbf{x}_2(t) = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}e^{3t}$$

$$= e^{3t}\left\{ -2\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \right\} = e^{3t}\begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix}$$

Check that $\mathbf{x}_3(t)$ is a solution:

$$\mathbf{x}_3'(t) \overset{?}{=} P\mathbf{x}_3(t)$$

$$\mathbf{x}_3'(t) = 5e^{5t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = e^{5t}\begin{bmatrix} 10 \\ -10 \\ 5 \end{bmatrix}$$

$$P\mathbf{x}_3(t) = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} e^{5t}$$

$$= \left\{ 2\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \right\} e^{5t}$$

$$= \begin{bmatrix} 10 \\ -10 \\ 5 \end{bmatrix} e^{5t}$$

Check that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are LI:

$$0 \neq \det |\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ \mathbf{x}_3(t)| = e^{t}e^{3t}e^{5t}\begin{vmatrix} 2 & -2 & 2 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= e^{9t}\left\{ -2\begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} \right\}$$

$$= e^{9t}\{-2(-4) - (-2)4\} = 16e^{9t}.$$

The Wronskian is never zero, so the solutions $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are LI.

General solution of $\mathbf{x}' = P\mathbf{x}$: $\quad \mathbf{x}(t) = A\mathbf{x}_1(t) + B\mathbf{x}_2(t) + C\mathbf{x}_3(t)$.