5.2 THE EIGENVALUE METHOD FOR LINEAR SYSTEMS

GOAL develop a method for solving ANY system of the form

$$(\mathrm{I})\ \begin{cases} x_1' = a x_1 + b x_2 \\ x_2' = c x_1 + d x_2 \end{cases} \qquad \mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}$$

WE KNOW How to solve any system with $a = 0$.

$$(\mathrm{II})\ \begin{cases} x_1' = x_2 \\ x_2' = c x_1 + d x_2 \end{cases} \qquad x_1'' - d x_1' - c x_1 = 0$$

ALSO KNOW how to solve any system with $b = c = 0$:

$$(\mathrm{III})\ \begin{cases} x_1' = a x_1 \\ x_2' = d x_2 \end{cases} \qquad \Rightarrow x_1(t) = A e^{at},\ x_2(t) = B e^{dt}$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{at} + B \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{bt}$$

IDEA find a change of variables that turns (I) into (III).

The matrix in (III) is diagonal, so the change of variables we are looking for probably is related with diagonalization of matrices.

EIGENVALUES AND EIGENVECTORS

DEFINITION Let P be a square $n \times n$ matrix. We say that the vector $\mathbb{v}$ (an $n \times 1$ matrix) is an **eigenvector** of P if

i) $\mathbb{v} \neq 0$

ii) $P\mathbb{v} = \lambda \mathbb{v}$ for some number λ, that is called the **eigenvalue** associated to λ.

EXAMPLES $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ Eigenvalues 1 and 4, eigenvectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$ Eigenvalues $3 \pm 4i$, eigenvectors $\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$.

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Eigenvalues $3, 0, 0$. eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

HOW TO SOLVE $\mathbf{x}' = P\mathbf{x}$ KNOWING THE EIGENVECTORS AND EIGENVALUES OF P.

EXAMPLE $\quad \mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \mathbf{x}$

Need two LI solutions.

We know $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^t = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^t = \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^t\right)'$

so $\mathbf{x}_1(t) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^t$ is a solution of $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \mathbf{x}$.

Also know $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}\right)'$

so $\mathbf{x}_2(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ is also a solution of $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \mathbf{x}$.

It's easy to see that $\mathbf{x}_1(t)$ and $\mathbf{x}_2$ are LI, but we can verify if the Wronskian is $\neq 0$, for practice:

$$0 \overset{?}{\neq} \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -1.$$

So the general solution of $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \mathbf{x}$ is $\boxed{\mathbf{x}(t) = Ae^t \begin{bmatrix} -1 \\ 2 \end{bmatrix} + Be^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

EXAMPLE $\mathbf{x}' = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} \mathbf{x}$

Need two LI solutions.

Know: $\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (3+4i) \begin{bmatrix} 1 \\ i \end{bmatrix}$

This gives a complex solution $\mathbf{z}(t) := e^{t(3+4i)} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{3t}(\cos 4t + i \sin 4t) \begin{bmatrix} 1 \\ i \end{bmatrix}$

(separate terms with i from the others) $= e^{3t} \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} + i e^{3t} \begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix}$

$=: \mathbf{x}_1(t) + i \mathbf{x}_2(t).$

Notice that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the equation $\mathbf{x}' = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} \mathbf{x}$ (you can check this directly or use the linearity of the matrix product).

They are also LI. That's clear because they are not constant multiples of each other, but we can also compute the Wronskian for practice:

$$0 \overset{?}{\neq} \begin{vmatrix} \cos 4t & \sin 4t \\ -\sin 4t & \cos 4t \end{vmatrix} = \cos^2 4t + \sin^2 4t = 1.$$

So the general solution is $\boxed{\mathbf{x}(t) = A e^{3t} \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} + B e^{3t} \begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix}}$.

WHAT ABOUT THE CHANGE OF VARIABLES?

We didn't really change variables in these two examples, but the substitution was there.

In the first example, $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\mathbf{x}$, the eigenvectors $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ give

$$\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

So if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ then $\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Since the matrix $\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$ is invertible, we get $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

In other words, the change of variables $x_1 = -y_1 + y_2$, $x_2 = 2y_1 + y_2$ transforms

$$\begin{cases} x_1' = 3x_1 + x_2 \\ x_2' = 2x_1 + 2x_2 \end{cases} \quad \text{into} \quad \begin{cases} y_1' = y_1 \\ y_2' = 4y_2. \end{cases}$$