5.2 THE EIGENVALUE METHOD

GOAL develop method to solve $\mathbf{x}' = P\mathbf{x}$, ($P$ is a square matrix)

KNOW .) If P is an $n \times n$ matrix, need n LI solutions.

..) If $P\mathbf{v} = \lambda \mathbf{v}$ and $\mathbf{v} \neq 0$ (i.e. $\mathbf{v}$ is an eigenvector)

then $e^{\lambda t}\mathbf{v}$ is a solution

$$\left(P(e^{\lambda t}\mathbf{v}) = e^{\lambda t} P\mathbf{v} = \lambda e^{\lambda t}\mathbf{v} = (e^{\lambda t}\mathbf{v})' \right)$$

If $\lambda = a+bi$, $b \neq 0$ then $\mathrm{Re}(e^{\lambda t}\mathbf{v})$ and $\mathrm{Im}(e^{\lambda t}\mathbf{v})$

are solutions.

TO DO Compute eigenvectors.

Deal with "repeated" eigenvalues.

CHARACTERISTIC POLYNOMIAL AND CHARACTERISTIC EQUATION

P is a $n \times n$ matrix, want to find its eigenvalues.

Want λ and $\mathbf{v}$ such that

$$P\mathbf{v} = \lambda \mathbf{v}$$

$$(P - \lambda I)\mathbf{v} = 0 \implies P - \lambda I \text{ is NOT invertible (i.e. columns are LD)}$$

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \implies \boxed{\det(P - \lambda I) = 0}$$

characteristic equation for P

EXAMPLES

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Characteristic polynomial

$$P - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} \qquad \det(P - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(4-\lambda) - 6$$

$$= \lambda^2 - 5\lambda - 2$$

Characteristic equation $0 = \lambda^2 - 5\lambda - 2$

EXAMPLE $y'' + 3y' + 2y = 0$

Characteristic eq: $r^2 + 3r + 2 = 0.$

In system form ($x_1 = y$, $x_2 = y'$)

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 3x_2 \end{cases} \qquad \mathbf{x}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{P} \mathbf{x}$$

Charact. eq. of P:

$$0 = \det \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = \lambda^2 + 3\lambda + 2$$

EXAMPLE $\begin{cases} x_1' = 3x_1 + 4x_2 \\ x_2' = 3x_1 + 2x_2 \end{cases}$ $x_1(0) = x_2(0) = 1$

In matrix form: $\mathbf{x}' = \underbrace{\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}}_{=:P} \mathbf{x}.$

Need two LI solutions.

① Look for eigenvalues.

The eigenvalues are the zeros of the characteristic equation

$$0 = \det(P - \lambda I) = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 12$$

$$= \lambda^2 - 5 - 6$$

$$= (\lambda + 1)(\lambda - 6)$$

$\boxed{\text{Eigenvalues } -1 \text{ and } 6.}$

③ Compute eigenvectors

To find ~~one~~ an eigenvector for -1, need to solve

$$P\mathbf{v} = -\mathbf{v} \Longrightarrow (P+I)\mathbf{v} = 0$$

$$\begin{bmatrix} 3+1 & 4 \\ 3 & 2+1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Infinitely many solutions, pick one, for example $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Check

$$\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So ~~one~~ a solution to $\mathbf{x}' = P\mathbf{x}$ is

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

To find eigenvector for 6, solve

$$\begin{bmatrix} 3-6 & 4 \\ 3 & 2-6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Pick one solution, for example $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

This yields $\mathbf{x}_2(t) = \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t}$

Eigenvector associated to 6:

$$\begin{bmatrix} 3-6 & 4 \\ 3 & 2-6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(P-6I)v = v$

$\iff Pv = 6v$

$$\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Pick for example $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, 6

(3) ~~Write~~ Find LI solutions to $x' = Px$ and write down general solution.

$$x_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \qquad x_2(t) = \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t}$$

General solution: $x(t) = A\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + B\begin{bmatrix} 4 \\ 3 \end{bmatrix}e^{6t}$

$$x_1(t) = Ae^{-t} + 4Be^{6t}$$

$$x_2(t) = -Ae^{-t} + 3Be^{6t}$$

(4) Use $x_1(0) = x_2(0) = 1$ to solve for A and B.

$$\begin{cases} 1 = A + 4B \\ 1 = -A + 3B \end{cases} \Rightarrow A = -\frac{1}{7}$$

$$B = \frac{2}{7}$$

ANSWER :

$$\boxed{\begin{aligned} x_1(t) &= -\frac{1}{7}e^{-t} + \frac{8}{7}e^{6t} \\ x_2(t) &= \frac{1}{7}e^{-t} + \frac{6}{7}e^{6t} \end{aligned}}$$

EXAMPLE $x_1' = x_1 - 5x_2$, $x_2' = x_1 - x_2$

$$\mathbb{x}' = \underbrace{\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}}_{P} \mathbb{x}$$

① Eigenvalues & vectors (via charact. eq.)

② Two LI solutions for $\mathbb{x}' = P\mathbb{x}$.

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① $0 = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)+5 = \lambda^2+4$

$= (\lambda+2i)(\lambda-2i)$

Eigenvalues $\pm 2i$

Eigenv. associated to $2i$ ($P\mathbb{v} = 2i\mathbb{v} \iff (P-2iI)\mathbb{v} = 0$)

$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Pick one eigenvect.

$\begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$, for example

② Get <u>complex</u> solution to $\mathbf{x}' = P\mathbf{x}$

$$\mathbf{z}(t) := \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} e^{2it}$$

$\Rightarrow$ Re $\mathbf{z}(t)$ and Im $\mathbf{z}(t)$ are real LI solutions of $\mathbf{x}' = P\mathbf{x}$.

$$\mathbf{z}(t) = \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} (\cos 2t + i \sin 2t)$$

$$= \underbrace{\begin{bmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix}}_{=:\mathbf{x}_1} + i \underbrace{\begin{bmatrix} 5 \sin 2t \\ \sin 2t - 2\cos 2t \end{bmatrix}}_{=:\mathbf{x}_2(t)}$$

General solution $\mathbf{x}(t) = A\mathbf{x}_1(t) + B\mathbf{x}_2(t)$

$$x_1(t) = 5A\cos 2t + 5B\sin 2t$$

$$x_2(t) = A(\cos 2t + 2\sin 2t) + B(\sin 2t - 2\cos 2t)$$

EXAMPLE $y'' + y' - 2y = 0$

$$\begin{cases} x_1' = x_2 \\ x_2' = 2x_1 - x_2 \end{cases} \qquad \mathbb{x}' = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \mathbb{x}$$

Eigenvalues -2 and $+1$

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

eigenvector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\mathbb{x}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

$$\mathbb{x}_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

$$\Rightarrow x_1(t) = Ae^{t} + Be^{-2t}$$