5.5 MULTIPLE EIGENVALUE SOLUTIONS

Consider $\mathbf{x}' = P\mathbf{x}$ where P is a 2×2 matrix with repeated eigenvalue λ.

Example: $P = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $\det(P-\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} =$

$= \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$

THEOREM: The solution to $\mathbf{x}' = P\mathbf{x}$ has the form

$$\mathbf{x}(t) = \mathbf{v}_0 e^{\lambda t} + \mathbf{v}_1 t e^{\lambda t},$$

where

$$(P-\lambda)\mathbf{v}_0 = \mathbf{v}_1 \checkmark$$

$$(P-\lambda)\mathbf{v}_1 = 0.$$

Check:

$$\mathbf{x}'(t) = \lambda \mathbf{v}_0 e^{\lambda t} + (1+\lambda t)\mathbf{v}_1 e^{\lambda t}$$

$$= e^{\lambda t}\left(\underline{\lambda \mathbf{v}_0 + \mathbf{v}_1}\right) + t e^{\lambda t}\left(\underline{\lambda \mathbf{v}_1}\right)$$

$$P\mathbf{x}(t) = e^{\lambda t}\left(\underline{P\mathbf{v}_0}\right) + t e^{\lambda t}\left(\underline{P\mathbf{v}_1}\right)$$

REMARK: $\mathbf{x}(0) = \mathbf{v}_0$

$$P\mathbf{v}_1 = \lambda \mathbf{v}_1$$

EXAMPLE:

$$\mathbf{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Compute eigenvalues

$$0 = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 16 = (\lambda-4)^2$$

Can look for a solution of the form

$$\mathbf{x}(t) = \mathbf{v}_0 e^{\lambda t} + \mathbf{v}_1 t e^{\lambda t}$$

$$\mathbf{v}_0 = \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_1 = (P-4)\mathbf{v}_0 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -9 \\ 9 \end{bmatrix}$$

$$\Rightarrow \boxed{\mathbf{x}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t} + \begin{bmatrix} -9 \\ 9 \end{bmatrix} t e^{4t}}$$

EXAMPLE $\mathbb{X}' = \underbrace{\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}}_{P} \mathbb{X}$

Eigenvalues : $0 = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$

Double eigenvalue $\Rightarrow$ look for $\mathbb{V}_0$ and $\mathbb{V}_1$ such that

$$\mathbb{X}(t) = \mathbb{V}_0 e^{3t} + \mathbb{V}_1 t e^{3t}$$

$$(P-3)\mathbb{V}_0 = \mathbb{V}_1$$

$$(P-3)\mathbb{V}_1 = 0$$

Two constants in the general solution, can choose the entries of $\mathbb{V}_0$.

$$\mathbb{V}_1 = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\mathbb{V}_0} = \begin{bmatrix} -2a - 2b \\ 2a + 2b \end{bmatrix}$$

$$\mathbb{X}(t) = \begin{bmatrix} a \\ b \end{bmatrix} e^{3t} + \begin{bmatrix} -2a-2b \\ 2a+2b \end{bmatrix} t e^{3t}$$

To find the general solution, let the entries of $\mathbf{v}_0$ be free parameters, i.e.

$$\mathbf{v}_0 = \begin{bmatrix} a \\ b \end{bmatrix}.$$

$$\mathbf{v}_1 = \underbrace{\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}}_{P-4} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\mathbf{v}_0} = \begin{bmatrix} -3a-3b \\ 3a+3b \end{bmatrix}$$

$$\Longrightarrow \boxed{\mathbf{x}(t) = \begin{bmatrix} a \\ b \end{bmatrix} e^{4t} + \begin{bmatrix} -3a-3b \\ 3a+3b \end{bmatrix} t e^{4t}}$$

EXAMPLE

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}}_{P} \mathbf{x}$$

Eigenvalues $\quad 0 = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^3$

So 2 is a triple eigenvalue.

Need 3 LI solutions to $\mathbf{x}' = P\mathbf{x}$.

If $\mathbf{v}_2$ is an eigenvector (i.e. $(P-2)\mathbf{v}_2 = 0$) then

$\mathbf{x}_2(t) = \mathbf{v}_2 e^{2t}$ is a solution

Guess $\quad \mathbf{x}_1(t) = \mathbf{v}_1 e^{2t} + \mathbf{v}_2 t e^{2t}$

What is $\mathbf{v}_1$?

Need $\quad \mathbf{x}_1' = P\mathbf{x}_1$

$$\mathbf{x}_1'(t) = 2\mathbf{v}_1 e^{2t} + (1+2t)e^{2t}\mathbf{v}_2$$

$$= e^{2t}(2\mathbf{v}_1 + \mathbf{v}_2) + te^{2t}(\cancel{2\mathbf{v}_2})$$

$$P\mathbf{x}_1(t) = (P\mathbf{v}_1)e^{2t} + (\cancel{P\mathbf{v}_2})te^{2t}$$

Since $\mathbf{x}_1'(t) = P\mathbf{x}_1(t)$, must have $(P-2)\mathbf{v}_1 = \mathbf{v}_2$

Have two LI solutions to $\mathbf{x}' = P\mathbf{x}$

$$\mathbf{x}_2(t) = \mathbf{v}_2 e^{2t}$$

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{2t} + \mathbf{v}_2 t e^{2t} \qquad (P-2)\mathbf{v}_1 = \mathbf{v}_2$$

Need a third solution

Guess: $\quad \mathbf{x}_0(t) = \mathbf{v}_0 e^{2t} + \mathbf{v}_1 t e^{2t} \quad \mathbf{v}_2 \frac{t^2}{2} e^{2t}$

What is $\mathbf{v}_0$? $\quad$ (in the end $(P-2)\mathbf{v}_0 = \mathbf{v}_1$)

$$\mathbb{x}_0(t) = e^{2t}\left(\mathbb{v}_0 + \mathbb{v}_1 t + \mathbb{v}_2 \frac{t^2}{2}\right)$$

$$\mathbb{x}_0'(t) = e^{2t}\left(\mathbb{v}_1 + 2\mathbb{v}_0 + \mathbb{v}_2 t + 2\mathbb{v}_1 t + \mathbb{v}_2 t^2\right)$$

$$(f(t)e^{2t})' = e^{2t}(f'(t) + 2f(t))$$

$$P\mathbb{x}_0(t) = e^{2t}\left(P\mathbb{v}_0 + (P\mathbb{v}_1)t + (P\mathbb{v}_2)\frac{t^2}{2}\right)$$

$$\mathbb{x}_0'(t) = P\mathbb{x}_0(t)$$

$$(\mathbb{v}_1 + 2\mathbb{v}_0) + (\cancel{\mathbb{v}_2 + 2\mathbb{v}_1})t + (\cancel{\mathbb{v}_2})t^2 =$$

$$= (P\mathbb{v}_0) + (\cancel{P\mathbb{v}_1})t + \left(\cancel{\tfrac{1}{2}\underbrace{P\mathbb{v}_2}_{2\mathbb{v}_2}}\right)t^2$$

$$(P-2)\mathbb{v}_1 = \mathbb{v}_2$$

$$\Longrightarrow \boxed{\begin{aligned} (P-2)\,\mathbb{V}_0 &= \mathbb{V}_1 \\ (P-2)\,\mathbb{V}_1 &= \mathbb{V}_2 \\ (P-2)\,\mathbb{V}_2 &= 0 \end{aligned}} \qquad \leftarrow \text{new eq}$$

For our matrix

$$\mathbb{V}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbb{V}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbb{V}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Check</u>: $(P-2)\,\mathbb{V}_1 \stackrel{?}{=} \mathbb{V}_2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$(P-2)\,\mathbb{V}_0 = \mathbb{V}_1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

GENERAL SOLUTION:

$$\begin{aligned} \mathbb{X}(t) = {} & A\,\mathbb{V}_2 e^{2t} \\ & + B\left(\mathbb{V}_1 + \mathbb{V}_2 t\right) e^{2t} \\ & + C\left(\mathbb{V}_0 + \mathbb{V}_1 t + \mathbb{V}_2 \frac{t^2}{2}\right) e^{2t} \end{aligned} = \begin{bmatrix} C e^{2t} \\ (B + Ct)e^{2t} \\ \left(A + Bt + \frac{Ct^2}{2}\right) e^{2t} \end{bmatrix}.$$