5.5 MULTIPLE EIGENVALUES

$\mathbf{x}' = P\mathbf{x}$, P square matrix

The eigenvalues and eigenvectors of P determine the solutions.

If $P\mathbf{v} = \lambda\mathbf{v}$ then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ solves $\mathbf{x}' = P\mathbf{x}$.

If $P\mathbf{v} = (a+ib)\mathbf{v}$, $b \neq 0$ then $\mathrm{Re}\, e^{(a+ib)t}\mathbf{v}$ and $\mathrm{Im}\, e^{(a+ib)t}\mathbf{v}$

are LI solutions.

EXAMPLE WITH MULTIPLE EIGENVALUES

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ -1 & 5 & 1 \\ 0 & 0 & 4 \end{bmatrix}}_{P} \mathbf{x}$$

Characteristic polynomial

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ -1 & 5-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 5-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda)^2(5-\lambda)$$

Eigenvalues 4, 4, 5.

Find eigenvectors to 4

$(P-4)\mathbb{v}_1 = \mathbb{0}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If $\mathbb{v}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $-x_1 + x_2 + x_3 = 0$

two free variables

$\Rightarrow$ two LI eigenvectors

Can pick LI solutions $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\Rightarrow$ $\mathbb{x}' = P\mathbb{x}$ has solutions $e^{4t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $e^{4t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

P is 3x3 $\Rightarrow$ we need 3 LI solutions

~~use~~ solve $(P-5)\mathbb{w} = \mathbb{0}$ ~~and~~ to get solution $e^{5t}\mathbb{w}$

In fact, $\mathbb{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

DEF Let P be a square matrix and λ an eigenvalue of P.

We say $\mathbb{v}_1$ is a <u>1-chain</u> if $(P-\lambda)\mathbb{v}_1 = 0$, and $\mathbb{v}_1 \neq 0$.

We say that $(\mathbb{v}_1, \mathbb{v}_2)$ is a <u>2-chain</u> if
$$(P-\lambda)\mathbb{v}_1 = 0, \quad \mathbb{v}_1 \neq 0.$$
$$(P-\lambda)\mathbb{v}_2 = \mathbb{v}_1$$

We say that $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3)$ is a <u>3-chain</u> if
$$(P-\lambda)\mathbb{v}_1 = 0, \quad \mathbb{v}_1 \neq 0.$$
$$(P-\lambda)\mathbb{v}_2 = \mathbb{v}_1$$
$$(P-\lambda)\mathbb{v}_3 = \mathbb{v}_2$$

~~So~~ USING CHAINS TO SOLVE $\mathbb{x}' = P\mathbb{x}$.

If $\mathbb{v}_1$ is a 1-chain then $\mathbb{x}_1(t) = e^{\lambda t}\mathbb{v}_1$ solves $\mathbb{x}' = P\mathbb{x}$.

If $(\mathbb{v}_1, \mathbb{v}_2)$ is a 2-chain then
$$\mathbb{x}_2(t) = t e^{\lambda t}\mathbb{v}_1 + e^{\lambda t}\mathbb{v}_2 \quad \text{solves} \quad \mathbb{x}' = P\mathbb{x}.$$

If $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3)$ is a 3-chain then
$$\mathbb{x}_3(t) = \frac{t^2}{2} e^{\lambda t}\mathbb{v}_1 + t e^{\lambda t}\mathbb{v}_2 + e^{\lambda t}\mathbb{v}_3$$

Let's check $\mathbb{x}_3'(t) = P\mathbb{x}_3(t)$

$$\mathbb{x}_3(t) = e^{\lambda t}\left(\frac{t^2}{2}\mathbb{v}_1 + t\,\mathbb{v}_2 + \mathbb{v}_3\right)$$

$$(e^{\lambda t} \cdot f(t))' = \\ = e^{\lambda t}(f'(t) + \lambda f(t))$$

$$\mathbb{x}_3'(t) = e^{\lambda t}\Big(t\mathbb{v}_1 + \mathbb{v}_2 + \cancel{\lambda \frac{t^2}{2}\mathbb{v}_1} + \lambda t\mathbb{v}_2 + \lambda\mathbb{v}_3\Big)$$

$$P\mathbb{x}_3(t) = e^{\lambda t}\left(\cancel{\frac{t^2}{2}P\mathbb{v}_1} + tP\mathbb{v}_2 + P\mathbb{v}_3\right)$$

$$P\mathbb{v}_3 = \lambda\mathbb{v}_3 + \mathbb{v}_2$$

$$P\mathbb{v}_2 = \lambda\mathbb{v}_2 + \mathbb{v}_1$$

EXAMPLE

$$\mathbf{x}' = \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}}_{P} \mathbf{x}$$

Characteristic polynomial

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & -1-\lambda & 1 \\ -1 & -1 & -1-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 0 & -1-\lambda \\ -1 & -1 \end{vmatrix}$$

$$= \dots = (-1-\lambda)^3$$

Eigenvalues $-1, -1, -1$.

Need 3 LI solutions to $\mathbf{x}' = P\mathbf{x}$.

-1 is a triple eigenvalue.

Several possibilities:

↪ three 1-chains (i.e. 3 LI eigenv.)

↪ one 1-chain, ~~&~~ one 2-chain

↪ one 3-chain

To find which, solve $(P+1)\mathbf{v}_1 = \mathbf{0}$.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{v_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} x_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

one free variabe

$\Rightarrow$ only one LI eigenvector

$\Rightarrow$ 3-chain

Can pick $v_1 = \begin{bmatrix} 1 \\ +1 \\ 0 \end{bmatrix}$

Look for 3-chain (v_1, v_2, v_3)

$\lambda = -1$

$$0 \leftarrow v_1 \leftarrow v_2 \xleftarrow{P-\lambda} v_3$$

$$(P+1)v_2 = v_1$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{v_2} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} \qquad \begin{aligned} x_3 &= \cancel{0}\ 1 \\ x_3 &= 1 \\ x_1 - x_2 &= \cancel{0} \end{aligned}$$

Pick $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$(P+I)\mathbf{v}_3 = \mathbf{v}_2$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$x_3 = 0$

$x_1 - x_2 = 1$

Pick ~~$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$~~

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

With the 3-chain $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, can construct 3 LI solutions.

$$\begin{array}{ccc} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{array}$$

$$\mathbf{x}_1(t) = e^t \mathbf{v}_1 = e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_2(t) = e^t (t\mathbf{v}_1 + \mathbf{v}_2) = e^t \begin{bmatrix} t \\ t \\ 1 \end{bmatrix}$$

$$\mathbf{x}_3(t) = e^t \left(\frac{t^2}{2}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\right) = e^t \begin{bmatrix} \frac{t^2}{2} + 1 \\ \frac{t^2}{2} \\ t \end{bmatrix}$$

$$e^t \begin{bmatrix} a + bt + c + c\frac{t^2}{2} \\ a + bt + ct^2/2 \\ b + ct \end{bmatrix}$$