## 3.6 FUNDAMENTAL MATRICES AND MATRIX EXPONENTIALS

The general solution of $\mathbf{x}' = P\mathbf{x}$ ($P$ an $n \times n$ matrix) has the form

$$\circledast \qquad \mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t), \quad \text{where} \quad \mathbf{x}_j'(t) = P\mathbf{x}_j(t) \quad \text{and} \quad \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \text{ LI}$$

Today: find compact formulas to express the solutions of $\mathbf{x}' = P\mathbf{x}$.

### Fundamental matrices

Rewrite $\circledast$ as a matrix product

$$\mathbf{x}(t) = \underbrace{\begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{bmatrix}}_{\Phi(t)} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

**DEFINITION** We say that a matrix-valued function $\Phi(t)$ is a **fundamental matrix** for $\mathbf{x}' = P\mathbf{x}$ if $\Phi(t)$ is square and

•) $\det \Phi(t) \neq 0$ for some $t$ (Wronskian determinant, Section 4.2)

••) $\Phi'(t) = P\Phi(t)$

$$\mathbf{x}(t) = \underbrace{\begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{bmatrix}}_{\Phi(t)} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

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**Formulas for $\mathbf{x}' = P\mathbf{x}$.**

The general solution of $\mathbf{x}' = P\mathbf{x}$ has the form $\mathbf{x}(t) = \Phi(t)\mathbf{c}$, where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

The solution of the IVP for $\mathbf{x}' = P\mathbf{x}$ (i.e. $\mathbf{x}(0)$ is Known) is

$$\boxed{\mathbf{x}(t) = \Phi(t)\,\Phi(0)^{-1}\,\mathbf{x}(0)} \leftarrow \mathbf{x}(0) = \Phi(0)\,\Phi(0)^{-1}\,\mathbf{x}(0) = \mathbf{x}(0)$$

**Remark:** there are many fundamental matrices for the same system $\mathbf{x}' = P\mathbf{x}$.
If $C$ is a square matrix with $\det C \neq 0$ then $\Phi(t)C$ is also a fundamental matrix.

EXAMPLE $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Notice that

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So two LI solutions are

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

$$\mathbf{x}_2(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$$

So a fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix}$$

Other fundamental matrix is

$$\Psi(t) = \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix}}_{\Phi(t)} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{C}$$

To solve the IVP, need to find a vector $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ such that $\Phi(0)\mathbf{c} = \mathbf{x}(0)$ (i.e. $\mathbf{c} = \Phi(0)^{-1}\mathbf{x}(0)$)

$$\Phi(0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

To compute $\Phi(0)^{-1}$, row reduce $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$ until the matrix on the left is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$-1\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right]\frac{1}{2} \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -2 & -1 & 1 \end{array}\right]\times\left(-\frac{1}{2}\right) \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

EXAMPLE $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Check $\Phi(0)\Phi(0)^{-1} = 1$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix} \qquad \mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}(0) \qquad \Phi(0)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boxed{\mathbf{x}(t) = \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix}\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}}$$

# MATRIX EXPONENTIALS

Recall if $\Phi(t)$ is a fundamental matrix for $\mathbf{x}' = P\mathbf{x}$ then

$$\mathbf{x}(t) = \Phi(t)\,\Phi(0)^{-1}\,\mathbf{x}(0)$$

We'll define a special matrix $e^{Pt}$ in such a way that $\boxed{\mathbf{x}(t) = e^{Pt}\,\mathbf{x}(0)}$.

Such matrix satisfies $e^{Pt} = \Phi(t)\,\Phi(0)^{-1}$ for any fundamental matrix $\Phi(t)$

**How to define / compute $e^{Pt}$ ?**

We know that $x'(t) = p\,x(t)$ has solution $x(t) = e^{pt}\,x(0)$.

Also know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

**DEFINITION** If $P$ is a square matrix, define

$$e^P = I + P + \frac{1}{2!}P^2 + \frac{1}{3!}P^3 + \frac{1}{4!}P^4 + \cdots$$

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

**FACT** The series always converges.

**DEFINITION** If $P$ is a square matrix, define

$$e^P = I + P + \frac{1}{2!}P^2 + \frac{1}{3!}P^3 + \frac{1}{4!}P^4 + \cdots$$

**EXAMPLE**

$$e^{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}, \text{ because}$$

$$e^{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix} + \cdots$$

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ax & by \\ az & bw \end{bmatrix}$$

$$= \begin{bmatrix} 1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \cdots & 0 \\ 0 & 1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \cdots \end{bmatrix}$$

**DEFINITION** If $P$ is a square matrix, define

$$e^{P} = I + P + \frac{1}{2!}P^2 + \frac{1}{3!}P^3 + \frac{1}{4!}P^4 + \cdots$$

**THEOREM (next lecture)** $\frac{d}{dt} e^{Pt} = P e^{Pt}$

This will lead to the formula $\mathbf{x}(t) = e^{Pt}\mathbf{x}(0)$ for the solution of $\mathbf{x}' = P\mathbf{x}$.