3.6 MATRIX EXPONENTIALS

Formulas for $\mathbf{x}' = P\mathbf{x}$. ($P$ is a square matrix)

1) **Fundamental matrix:** $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}(0)$, where $\Phi(t)$ is a square matrix whose columns are LI solutions of $\mathbf{x}' = P\mathbf{x}$.

2) **Matrix Exponentials:** $\mathbf{x}(t) = e^{Pt}\mathbf{x}(0)$, where e^{Pt} is the matrix defined by

$$e^{Pt} := I + Pt + P^2\frac{t^2}{2!} + P^3\frac{t^3}{3!} + P^4\frac{t^4}{4!} + \cdots$$

Compare with

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

COMPUTATION OF e^{Pt} IN SOME EASY CASES

A) P is a diagonal matrix

$$e^{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} t} = \begin{bmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{ct} \end{bmatrix}$$

B) P is nilpotent matrix (i.e. $P^n = 0$ for some n), for example if one of the columns is $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

C) $P = \lambda + N$

λ: $\begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ & & \ddots \\ 0 & & \lambda \end{bmatrix}$

N: nilpotent

D) In the remaing cases, use $e^{Pt} = \Phi(t)\,\Phi(0)^{-1}$.

B) P is nilpotent matrix (i.e. $P^n = 0$ for some n), for example if one of the columns is $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$e^{\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t} = ?$$

$$P = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{Pt} = I + Pt + P^2 \frac{t^2}{2!}$$

$$e^{Pt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 3t \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^2/2! \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & 3t + t^2/2! \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\begin{bmatrix}0&1\\0&0\end{bmatrix}t} = \begin{bmatrix}1&t\\0&1\end{bmatrix}$$

$$P = \begin{bmatrix}0&1\\0&0\end{bmatrix} \qquad P^2 = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow e^{Pt} = \begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}0&t\\0&0\end{bmatrix}$$

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c) $P = \lambda + N$ (multiple of identity plus nilpotent)

$$e^{\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}t} = ?$$

$$\underbrace{\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}}_{P} = \underbrace{\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}}_{\lambda I} + \underbrace{\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}}_{N}$$

For numbers, $e^{a+b} = e^a e^b$.

For matrices, not always.

If $e^{A+B} = e^A e^B$ then

$$e^A e^B = e^B e^A$$

For matrices, it can happen that $e^A e^B \neq e^B e^A$.

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$e^{At} e^{Bt} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^{2t} \end{bmatrix}$$

$$e^{Bt} e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} = \begin{bmatrix} e^t & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

Therefore $e^{(A+B)t} \neq e^{At} e^{Bt}$.

THEOREM: If $AB = BA$ (i.e. A and B commute) then

$$e^{(A+B)t} = e^{At} e^{Bt}.$$

Back to

$$e^{\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} t} = ?$$

λI commutes with ALL matrices

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} t} = e^{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} t} e^{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t}$$

$$= \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} e^{\lambda t}$$

<u>THEOREM:</u> $\frac{d}{dt} e^{Pt} = P e^{Pt}$ (so $e^{Pt} \mathbf{x}(0)$ solves $\mathbf{x}'(t) = P \mathbf{x}(t)$).