5.7 NONHOMOGENEOUS LINEAR SYSTEMS

Methods for $\mathbf{x}' = P\mathbf{x} + \mathbf{f}(t)$, $\mathbf{f}$ is a vector-valued function of t

↳ undetermined coefficients

↳ variation of parameters

EXAMPLE Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$$

By linearity, $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$

solution to $\mathbf{x}_h' = P\mathbf{x}$

any solution

UNDETERMINED COEFFICIENTS

STEP 1 : solve $\mathbf{x}_h' = P\mathbf{x}$

Eigenvalues and eigenvectors

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow x_h(t) = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

STEP 2 : Guess a particular solution (no ~~redunc~~ overlap with x_h)

$$x_p(t) = \begin{pmatrix} a \\ b \end{pmatrix} e^{t}$$

Plug into $x_p'(t) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x_p(t) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$ and solve for $\begin{pmatrix} a \\ b \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} a e^{t} \\ b e^{t} \end{pmatrix} + \begin{pmatrix} 2e^{t} \\ -e^{t} \end{pmatrix} = \begin{pmatrix} a e^{t} \\ b e^{t} \end{pmatrix}$$

$$\left.\begin{aligned} (b+2)\cancel{e^{t}} &= a\cancel{e^{t}} \\ (-2a-3b-1)\cancel{e^{t}} &= b\cancel{e^{t}} \end{aligned}\right| \quad \begin{cases} -a + b = -2 & \text{(I)} \\ -2a - 4b = 1 & \text{(II)} \end{cases}$$

$-2(\mathrm{I}) + (\mathrm{II})$: $\quad -6b = 5 \qquad b = -\frac{5}{6}$

$$-a - \frac{5}{6} = -2 \implies a = \frac{7}{6}$$

$\implies$ General solution:

$$x(t) = \underbrace{c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}}_{x_h} + \underbrace{\frac{1}{6} \begin{pmatrix} 7 \\ -5 \end{pmatrix} e^{t}}_{x_p}$$

Variation of parameters

Want a solution of $\mathbf{x}'(t) = P\mathbf{x}(t) + \mathbf{f}(t)$.

Know that if $\Phi(t)$ is a fundamental matrix then

$$(\Phi(t)\mathbf{v})' = P\,\Phi(t)\mathbf{v} \qquad \text{(no forcing)}$$

Guess $\boxed{\mathbf{x}_p(t) = \Phi(t)\mathbf{v}(t)}$,

Plug into the equation

$$(\underbrace{\Phi(t)}_{\text{matrix}}\underbrace{\mathbf{v}(t)}_{\text{vector}})' = P(\Phi(t)\mathbf{v}(t)) + \mathbf{f}(t)$$

$$\cancel{\Phi'(t)\mathbf{v}(t)} + \Phi(t)\mathbf{v}'(t) = \cancel{P\,\Phi(t)\mathbf{v}(t)} + \mathbf{f}(t)$$

Solve for $\mathbf{v}'(t)$ in $\Phi(t)\mathbf{v}'(t) = \mathbf{f}(t)$, then plug $\mathbf{v}$ into $\mathbf{x}_p(t) = \Phi(t)\mathbf{v}(t)$.

EXAMPLE $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{x} + \underbrace{\begin{pmatrix} 0 \\ 4e^t \end{pmatrix}}_{\mathbf{f}(t)}$ (find a particular solution)

VARIATION OF PARAMETERS

STEP 1 : Find a fundamental matrix $\Phi(t)$

STEP 2 : Solve for $\mathbf{v}'(t)$ in $\Phi(t)\mathbf{v}'(t) = \mathbf{f}(t)$

STEP 3 : $\mathbf{x}_p(t) = \Phi(t)\mathbf{v}(t)$.

STEP 1 : First, find eigenvalues and eigenvectors

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

eigenvalue 1

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

eigenvalue 3

$$\Rightarrow \Phi(t) = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix}$$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^t \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}$

STEP 2: Solve for $\mathbf{v}'(t) = \begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix}$ in

$$\Phi(t)\,\mathbf{v}'(t) = f(t)$$

$$\underbrace{\begin{pmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{pmatrix}}_{\Phi(t)} \begin{pmatrix} v_1'(t) \\ v_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 4e^{t} \end{pmatrix}$$

$$e^{t} v_1'(t) \quad e^{3t} v_2'(t) = 0$$

$$-e^{t} v_1'(t) \quad e^{3t} v_2'(t) = 4e^{t}$$

$\cancel{2 v_1'(t)}$

$$2e^{3t} v_2'(t) = 4e^{t}$$

$$\Rightarrow \boxed{v_2'(t) = 2e^{-2t}}$$

$$\rightarrow e^{t} v_1'(t) = -e^{3t}(2e^{-2t})$$

$$\Rightarrow \boxed{v_1'(t) = -2}$$

STEP 3: $\mathbb{x}_p(t) = \Phi(t)\,\mathbb{v}(t)$ (pick a $\mathbb{v}(t)$ based on $\mathbb{v}'(t)$)

$$v_1'(t) = -2$$

$$v_2'(t) = 2e^{-2t}$$

Can pick $\mathbb{v}(t) = \begin{bmatrix} -2t \\ -e^{-2t} \end{bmatrix}$.

$$\mathbb{x}_p(t) = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix}\begin{pmatrix} -2t \\ -e^{-2t} \end{pmatrix} = \begin{pmatrix} -2te^t - e^t \\ 2te^t - e^t \end{pmatrix}$$

$$\boxed{\mathbb{x}_p(t) = \begin{pmatrix} -2t-1 \\ 2t-1 \end{pmatrix} e^t}$$

SAME EXAMPLE $\quad \mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 4e^t \end{pmatrix}$ find particular solution

UNDETERMINED COEFFICIENTS

STEP 1: solve homogeneous eq.

$$x_h(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

STEP 2: guess $\mathbf{x}_p(t)$.

1st guess: $\mathbf{x}_p(t) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t$ overlaps with x_h won't work

2nd guess: $\mathbf{x}_p(t) = \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^t$

STEP 3: plug into the equation $\quad (g(t)e^t)' = (g'(t) + g(t))e^t$

$$x_p'(t) = \begin{pmatrix} b_1 t + a_1 + b_1 \\ b_2 t + a_2 + b_2 \end{pmatrix} e^t$$

... 4 linear eqs for a_1, a_2, b_1, b_2