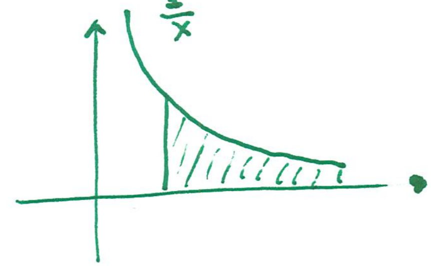
7.1 - 7.5 LAPLACE TRANSFORMS I

The LAPLACE TRANSFORM takes as input a function $f(t)$, $t \geq 0$ and outputs a function $F(s)$ defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$, defined for the values of s for which the integral converges

$\int_0^\infty 1\, dt = \infty$



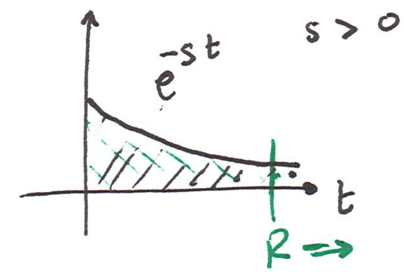
CONVENTIONS: ↪ inputs are functions of t
$f(t)$, $g(t)$, $y(t)$, ...

↪ outputs are functions of s
$F(s)$, $G(s)$, $Y(s)$, ...

↪ output also denoted $\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{1\} = \ ?$$

$\mathcal{L}\{1\}$ is a function of s



$$\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\, 1 \, dt$$

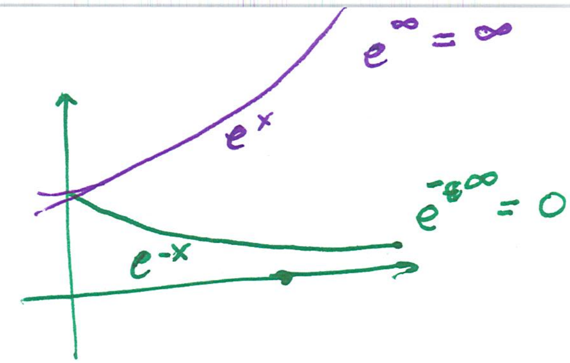
(by definition) $= \lim\limits_{R\to\infty} \int_0^R e^{-st}\, dt$

$$= \lim_{R\to\infty} \left. \frac{e^{-st}}{-s} \right|_0^{t=R}$$

$$= \lim_{R\to\infty} \frac{e^{-sR} - 1}{-s} = \begin{cases} \dfrac{1}{s}, & s > 0 \\ +\infty, & s < 0 \end{cases}$$

Therefore, $\boxed{\mathcal{L}\{1\} = \dfrac{1}{s}, \ s > 0}$

$$\mathcal{L}\{e^{at}\} = ?$$



$$\mathcal{L}\{e^{at}\}(s) = \int_0^{\infty} e^{-st} e^{at}\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\, dt$$

$$= \left. \frac{e^{-(s-a)t}}{s-a} \right|_0^{t=\infty}$$

$$(\text{if } s > a) \quad = \frac{1}{s-a}, \quad s > a$$

Remark: If $a = 0$, $e^{0t} = 1$, $\mathcal{L}\{e^{0t}\} = \dfrac{1}{s-0}$, $s > 0$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$$

$\mathcal{L}\{e^{ct} f(t)\} = ?$ (in terms of $F(s)$)

$$\mathcal{L}\{e^{ct} f(t)\}(s) = \int_0^\infty e^{-st} e^{ct} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-c)t} f(t)\, dt = F(s-c),$$ $s-c$ in the domain of F

$\mathcal{L}\{f(t)+g(t)\} = F(s) + G(s)$, s ~~ith~~ in the domains of both F and G

$\mathcal{L}\{a f(t)\} = a F(s)$

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$, $s>0$, s in the domain of F

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)\, dt \underset{\uparrow}{=} e^{-st} f'(t)\Big|_0^{t=\infty} - \int_0^\infty -s e^{-st} f(t)\, dt$$

integration by parts (product rule)

$$= -f(0) + sF(s)$$

EXAMPLE $y'(t) + y(t) = e^t \quad , \quad y(0) = 0.$

Take transforms of both sides

$$\mathcal{L}\{y'(t) + y(t)\} = \mathcal{L}\{e^t\}$$

$$\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = \frac{1}{s-1} \quad , \quad s > 1$$

$$s\,Y(s) - \cancel{y(0)} + Y(s)$$

$$Y(s) = \frac{1}{(s-1)(s+1)}$$

$$y(t) = ?$$

KNOW: given $f(t)$, find $F(s)$

ACTUALLY NEED: given $Y(s)$, find $y(t)$.

THEOREM: If $F(s) = G(s)$ for all large s * and f and g are continuous. Then $f = g$.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$ ✓
e^{at}	$\frac{1}{s-a}, \quad s > a$ ✓
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$ ✓
$t^p \ (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$e^{ct} f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right), \ c > 0$
$\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f'(t)$	$sF(s) - f(0)$ ✓
$-t f(t)$	$F'(s)$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$\sinh at$	$\dfrac{a}{s^2 - a^2}$
$\cosh at$	$\dfrac{s}{s^2 - a^2}$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$u_c(t)$	$\dfrac{e^{-cs}}{s}$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$