LAPLACE TRANSFORMS II

$$\underset{t \geq 0}{f(t)} \longrightarrow \boxed{\quad \mathcal{L} \quad} \longrightarrow : \; F(s) \text{ or } \mathcal{L}\{f(t)\}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

EXAMPLE $\quad y'(t) + y(t) = e^t, \quad \underline{y(0) = 0}$

$$\mathcal{L}\{y'(t) + y(t)\} = \mathcal{L}\{e^t\} = \frac{1}{s-1}$$

$$s\,Y(s) - \cancel{y(0)} + Y(s)$$

$$Y(s) = \frac{1}{(s-1)(s+1)} \qquad y(t) = ?$$

$\dfrac{1}{(s-1)(s+1)}$ not on the table, but $\dfrac{1}{s-1}$ and $\dfrac{1}{s+1}$ are

$$\mathcal{L}\{e^t\} = \frac{1}{s-1} \qquad \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

Partial fractions:

$$\frac{1}{(s-1)(s+1)} = \frac{\frac{1}{2}}{s-1} - \frac{\frac{1}{2}}{s+1}$$

$$\overset{\shortparallel}{Y(s)} \qquad\qquad \overset{\shortparallel}{\mathcal{L}\left\{\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right\}}$$

Since $y(t)$ and $\frac{1}{2}(e^{t}-e^{-t})$ are both continuous and their transforms coincide, $\boxed{y(t) = \frac{1}{2}(e^{t}-e^{-t})}$.

QUICK REVIEW OF PARTIAL FRACTIONS

A quotient of two polynomials, $\frac{p(s)}{q(s)}$, can be split into a sum of "building blocks".

Building blocks are of two types:

1) $\frac{c}{s-a}$, $\frac{c}{(s-a)^2}$, $\frac{c}{(s-a)^3}$, ... (when $q(s)$ factors into linear factors)

2) $\frac{cs+d}{(s-a)^2+b}$, $\frac{cs+d}{((s-a)^2+b)^2}$, ... $b \neq 0$

EXAMPLE

$$\frac{5s}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$ for some A, B, C

To find A, B, C, multiply out and plug ~~adequate~~ convenient values for s

$$5s \cancel{(s-1)(s^2+1)} = A(s^2+1) + (Bs+C)(s-1)$$

Plug $s=1$: $\cancel{0 = 2A}$ $5 = 2A \Rightarrow A = \frac{5}{2}$

Plug $s=0$: $\quad 0 = \frac{5}{2} + C(-1) \Longrightarrow C = \frac{5}{2}$

Plug $s=2$: $\quad 10 = \frac{5}{2}\cdot 5 + \left(2B + \frac{5}{2}\right)\cdot 1$

$\cancel{\Longrightarrow B =}$

$$\frac{6s+1}{(s-1)^3(s^2+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{Ds+E}{s^2+3}$$

EXAMPLE $\quad y'' + 6y' + 34y = 0\ ; \quad y(0) = 3, \quad y'(0) = 1.$

Key formula: $\mathcal{L}\{y'(t)\} = s\,Y(s) - y(0)$

$= s\,Y(s) - 3$ **

$$\mathcal{L}\{y''(t)\} = s\left(s\,Y(s) - 3\right) - y'(0)$$
$$= s^2 Y(s) - 3s - 1 \quad *$$

$$\mathcal{L}\{y'' + 6y' + 34y\} =$$

$$= s^2 Y(s) - 3s - 1 + 6\,(s\,Y(s) - 3) + 34\,Y(s) = 0$$

$*$ $**$

Solve for $Y(s)$ then find $y(t)$ (partial fractions + table)

$$Y(s)\,(s^2 + 6s + 34) = 3s + 19$$

$$Y(s) = \frac{3s + 19}{s^2 + 6s + 34} = \mathcal{L}\{\,?\,\}$$

$$\frac{3s + 19}{s^2 + 6s + 34} = \frac{3s}{(s+3)^2 + 5^2} + \frac{19}{(s+3)^2 + 5^2} = \mathcal{L}\{\,?\,\}$$

~~roots s = -6s~~

Complete the square

$$s^2 + 6s + 9 + 25 = (s+3)^2 + 5^2$$

$\mathcal{L}\{\sin at\} = ?$

Know: $\mathcal{L}\{e^{bt}\} = \dfrac{1}{s-b}$, $s > a$

also works for complex b. Choose $b = ai$

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s-ai} = \frac{s+ai}{(s-ai)(s+ai)} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$\parallel$

$$\mathcal{L}\{\cos at + i \sin at\}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

NEXT: $\mathcal{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$