LAPLACE TRANSFORMS III

$$\underset{t \geq 0}{f(t)} \longrightarrow \boxed{\;\mathcal{L}\;} \longrightarrow F(s) \quad \text{or} \quad \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

because $\displaystyle\int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$

EXAMPLE $y'' + 6y' + 34y = 0$; $y(0) = 3$, $y'(0) = 1$.

Transform both sides.

Key formula: $\mathcal{L}\{y'\} = s\,\overbrace{Y(s)}^{\mathcal{L}\{y(t)\}} - y(0) = sY(s) - 3$

$$\Rightarrow \mathcal{L}\{y''\} = s\,(sY(s) - 3) - y'(0) = s^2 Y(s) - 3s - 1$$

$$\mathcal{L}\{y'' + 6y' + 34y\} = 0$$

$$[s^2 Y(s) - 3s - 1] + 6[s\,Y(s) - 3] + 34\,Y(s) = 0$$

$$\cdots$$

$$Y(s) = \frac{3s + 19}{s^2 + 6s + 34} = \mathcal{L}\{\,?\,\}$$

Compare with

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2 + b^2}$$

$$Y(s) = \frac{3(s+3) + 2\cdot 5}{(s+3)^2 + 5^2} = \mathcal{L}\{3e^{-3t}\cos 5t + 2e^{-3t}\,\cancel{\cos}\sin 5t\}$$

$a = -3 \quad b = 5$

So $\boxed{y(t) = \cancel{3}\,e^{-3t}(3\cos 5t + 2\sin 5t)}$

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$$

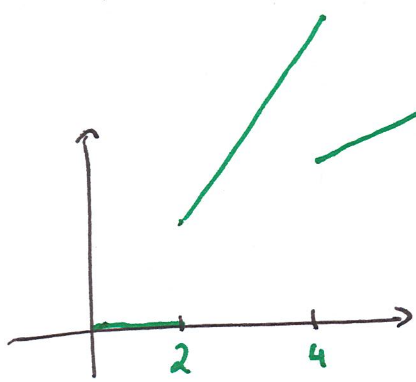
$$\mathcal{L}\{\ ?\ \} = e^{-cs}F(s)$$

Step functions can be used to express piecewise defined functions.

EXAMPLE

$$f(t) = \begin{cases} 0, & 0 \le t < 2 \\ 2t, & 2 \le t < 4 \\ t+1, & 4 \le t \end{cases}$$

$$f(t) = 2t\,(u_2(t) - u_4(t)) + (t+1)u_4(t)$$

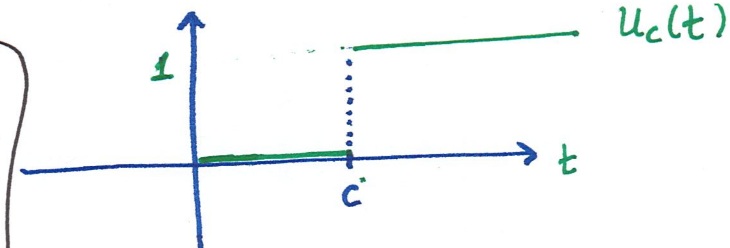


STEP FUNCTIONS

DEFINITION $c > 0$

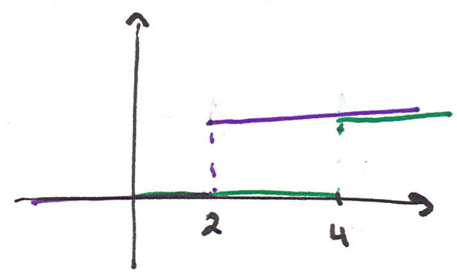
Define the HEAVISIDE FUNCTION by

$$u_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \ge c \end{cases}$$



Notice

$$u_2(t) - u_4(t) = \begin{cases} 1, & 2 \le t < 4 \\ 0, & \text{otherwise} \end{cases}$$

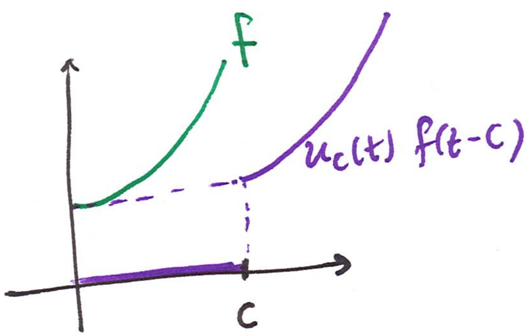


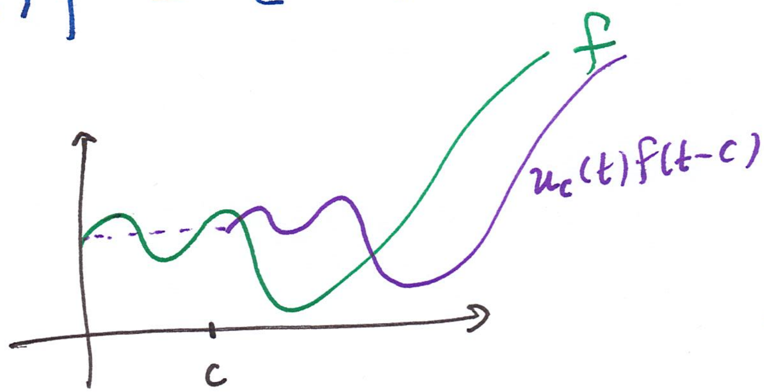
EXAMPLE Write $f(t) = \begin{cases} t, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ t-2, & 2 \le t \end{cases}$ in terms of step functions.

$$f(t) = t(1 - u_1(t)) + (t-1)(u_1(t) - u_2(t)) + (t-2)u_2(t)$$

FORMULA

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$$





$$+\int_0^c e^{-st}\, 0\, f(t-c)\, dt$$

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt = \int_c^\infty e^{-st} \cdot 1\, f(t-c)\, dt$$

$t - c = u$

$$= \int_0^\infty e^{-s(c+u)} f(u)\, du = e^{-sc} F(s)$$

EXAMPLE

$$y' = -y + u_3(t) \quad , \quad y(0) = 2$$

Transform, Solve for $Y(s)$, go back.

$$s\,Y(s) - 2 = -Y(s) + \frac{e^{-3s}}{s}$$

$$Y(s) = \underbrace{\frac{2}{s+1}}_{\mathcal{L}\{2e^{-t}\}} + \frac{e^{-3s}}{s(s+1)} = \mathcal{L}\{\,?\,\}$$

$$e^{-3s}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

Know. $\frac{1}{s} = \mathcal{L}\{1\}$

$$e^{-cs}F(s) = \mathcal{L}\{u_c(t)f(t-c)\}$$

$$\Longrightarrow \quad e^{-3s}\frac{1}{s} = \mathcal{L}\{u_3(t)\cdot 1\}$$

$$e^{-3s}\frac{1}{s+1} = \mathcal{L}\{u_3(t)e^{-(t-3)}\}$$