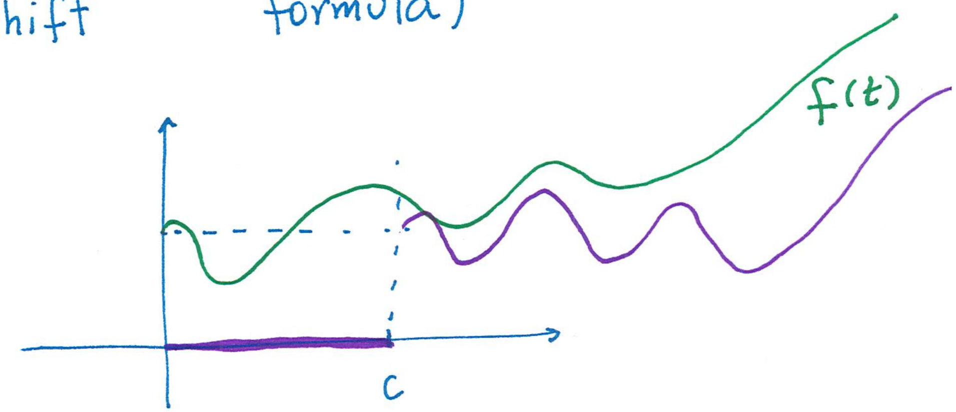
LAPLACE TRANSFORMS IV

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$$

(shift exponential formula)

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

"delay f by c"



$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{-t f(t)\} = F'(s)$$

Justification for $\mathcal{L}\{-t f(t)\}$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt = \int_0^\infty e^{-st}(-t f(t))\,dt$$

EXAMPLE $\mathcal{L}\{t \sin t\} = ?$

$$\mathcal{L}\{-t(-\sin t)\} = \frac{d}{ds}\mathcal{L}\{-\sin t\}$$

$t \sin 2t = t 2 \sin t \cos t$

$$= \frac{d}{ds}\frac{-1}{s^2+1}$$

$$= \frac{+2s}{(s^2+1)^2}$$

$\mathcal{L}\{-t f(t)\} = F'(s)$

EXAMPLE $\mathcal{L}\left\{ \underbrace{\frac{1-\cos 2t}{t}}_{f(t)} \right\} = ?$

We don't know $\mathcal{L}\{f(t)\}$ but we do know $\mathcal{L}\{-t f(t)\}$.

$$\mathcal{L}\{-t f(t)\} = \mathcal{L}\{-1 + \cos 2t\} = -\frac{1}{s} + \frac{s}{s^2+4}$$

Also $\mathcal{L}\{-t f(t)\} = F'(s)$

So $F'(s) = \frac{s}{s^2+4} - \frac{1}{s} \Rightarrow F(s) = \frac{\ln(s^2+4)}{2} - \ln(s) + C$

To find C, use $\lim_{s\to\infty} F(s) = 0$ because $F(s)=\int_0^\infty e^{-st}f(t)\,dt \xrightarrow{s\to\infty} 0$

$$F(s) = \frac{1}{2}\ln\left(\frac{s^2+4}{s^2}\right) + C$$

$$\frac{1}{2}\ln\left(\frac{s^2+4}{s^2}\right) = \frac{\ln(s^2+4)}{2} - \frac{1}{2}\ln(s^2)$$

$$\frac{1}{2}\ln\left(\frac{s^2+4}{s^2}\right) \xrightarrow{s\to\infty} 1$$

So $C=0$, and $\boxed{\mathcal{L}\left\{\frac{1-\cos 2t}{t}\right\} = \frac{1}{2}\ln\left(\frac{s^2+4}{s^2}\right)}$

EXAMPLE $\mathcal{L}^{-1}\left\{\ln\frac{s-2}{s+2}\right\} = ?$

We look for a function $f(t)$ such $F(s) = \ln\frac{s-2}{s+2}$.

Notice that $F'(s) = \frac{1}{s-2} - \frac{1}{s+2} = \mathcal{L}\{e^{2t} - e^{-2t}\}$

Also $F'(s) = \mathcal{L}\{-t\,f(t)\}$

So $-t\,f(t) = e^{2t} - e^{-2t}$

$$\boxed{f(t) = -\frac{e^{2t}-e^{-2t}}{t}}$$

EXAMPLE $\mathcal{L}^{-1}\{\tan^{-1}(\frac{1}{s})\}$

$\arctan'(u) = \frac{1}{1+u^2}$

Want $f(t)$ such that $F(s) = \tan^{-1}(\frac{1}{s})$.

$$F'(s) = \frac{1}{1+(\frac{1}{s})^2} \cdot \left(-\frac{1}{s^2}\right)$$

$$= \frac{-1}{s^2+1} = \mathcal{L}\{-\sin t\}$$

$$F'(s) = \mathcal{L}\{-t\, f(t)\}$$

$$\Longrightarrow \quad -t\, f(t) = -\sin t$$

$$\boxed{f(t) = \frac{\sin t}{t}}$$

CONVOLUTIONS

Given $f(t)$ and $g(t)$, $t \geq 0$. Define a function $(f*g)(t)$, $t \geq 0$ (the CONVOLUTION of f and g) by the formula

$$(f*g)(t) = \int_0^t f(\underset{\substack{\uparrow \\ \text{tau}}}{\tau})\, g(t-\tau)\, d\tau$$

Compare the integral with

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots) =$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2$$

$$+ (\underbrace{a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0}_{\text{“looks like” } \int_0^3 a(\tau) b(3-\tau)\, d\tau}) x^3 + \dots$$

$$\mathcal{L}\{(f*g)(t)\} = F(s)\,G(s)$$