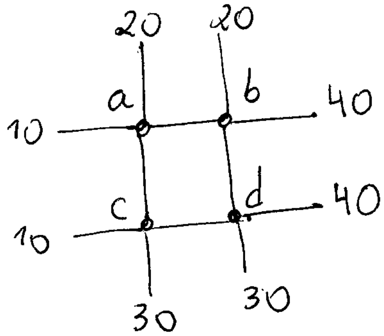
Lecture 01:

Problem: find a, b, c, d ~~x, y, z, w~~ given that each number is the average of its neighbors



Easier problem: find x and y such that

$$\begin{cases} x + 2y = 7 & (\mathrm{I}) \\ 2x + 3y = 11 & (\mathrm{II}) \end{cases}$$

Solution 1: solve for x in (I), plug into (II)

(I) $x = 7 - 2y$

(II) $2(7-2y) + 3y = 11$

Solution 2: eliminate variables by combining the eqs.

(II) $- 2$(I) : $-y = -3$

Gaussian elimination:

$$\begin{cases} x + 2y = 7 & (\mathrm{I}) \\ 2x + 3y = 11 & (\mathrm{II}) \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 3 & 11 \end{array}\right] -2$$

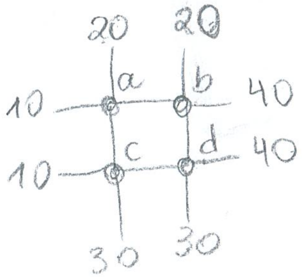
Replace (II) by (II) -2(I)

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 2-2\cdot 1 & 3-2\cdot 2 & 11-7\cdot 2 \end{array}\right]$$

Exercise: convince yourself that Solutions 1, 2 and 3 are exactly the same.

→ we replace a system with an equivalent system, the same way we do with ordinary equations.

Back to the first problem:



$$4a = 20 + b + c + 10$$
$$4b = 20 + 40 + d + a$$
$$4c = a + d + 30 + 10$$
$$4d = b + 40 + 30 + c$$

Convert into a system where all a's, b's, c's and d's are on the same columns.

$$\begin{aligned}
4a - b - c \phantom{{}-d} &= 30 \\
-a + 4b \phantom{{}-c} - d &= 60 \\
-a \phantom{{}-b} + 4c - d &= 40 \\
-b - c + 4d &= 70
\end{aligned}$$

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 30 \\ -1 & 4 & 0 & -1 & 60 \\ -1 & 0 & 4 & -1 & 40 \\ 0 & -1 & -1 & 4 & 70 \end{array}\right]$$

clear 1st column using (3,1) as a pivot

$30 + 4 \cdot 40$

$$\left[\begin{array}{cccc|c} 0 & -1 & 15 & -4 & 190 \\ 0 & 4 & -4 & 0 & 20 \\ -1 & 0 & 4 & -1 & 40 \\ 0 & -1 & -1 & 4 & 70 \end{array}\right]$$

clear 2nd column using row 4 as pivot

what happens if you clear the columns in a different order?

B

$$\left[\begin{array}{cccc|c} 0 & 0 & 16 & -8 & 120 \\ 0 & 0 & -8 & 16 & 300 \\ -1 & 0 & 4 & -1 & 40 \\ 0 & -1 & -1 & 4 & 70 \end{array}\right] \begin{array}{l} \leftarrow \cdot \frac{1}{-8} \\ \leftarrow \cdot -\frac{1}{8} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 0 & 0 & -2 & 1 & -15 \\ 0 & 0 & 1 & -2 & -37.5 \\ -1 & 0 & 4 & -1 & 40 \\ 0 & -1 & -1 & 4 & 70 \end{array}\right] \begin{array}{l} 2 \\ \\ -4 \\ +1 \end{array}$$

clear 3rd column using row 2 as pivot. (what goes wrong if row 4 is used as pivot ?)

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & -3 & -90 \\ 0 & 0 & 1 & -2 & -37.5 \\ -1 & 0 & 0 & 7 & 190 \\ 0 & -1 & 0 & 2 & 32.5 \end{array}\right]$$

$(-37.5)(-4)+40$

clear 4th column using row 1 as pivot. Divide by -3 before clearing.

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & -2 & -37.5 \\ 1 & 0 & 0 & -7 & -190 \\ 0 & 1 & 0 & -2 & -32.5 \end{array}\right] \begin{array}{l} \\ +2 \\ +7 \\ +2 \end{array}$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & 22.5 \\ 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 27.5 \end{array}\right] \longleftrightarrow \boxed{\begin{array}{l} a = 20 \\ b = 27.5 \\ c = 22.5 \\ d = 30 \end{array}}$$

If you clear rows instead of columns, what are you doing to the system of equations?

Lecture ~~3~~ 2B:

~~Systems with full solutions~~

Systems with unique solution.

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \phantom{{}+2w} &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

$$\begin{aligned} u + v + w &= 5 \\ 2u + 2v + 5w &= 3 \\ 4u + 6v + 8w &= 4 \end{aligned}$$

$$\begin{aligned} a + b + c + d &= 1 \\ a + 2b + 3c + 4d &= 5 \\ a + 3b + 6c + 9d &= 12 \\ a + 4b + 8c + 13d &= 19 \end{aligned}$$

Systems with infinitely many solutions

$$\begin{aligned} x + y + 3z + 3w &= 8 \\ 2x + 2y + 7z + 7w &= 18 \\ x + y \phantom{{}+7z} + w &= 3 \end{aligned}$$

GAUSSIAN ELIMINATION, FORWARD PHASE

INPUT : $m \times n$ matrix $(a_{ij})_{1 \le i \le m,\, 1 \le j \le n}$

OUTPUT : $m \times n$ row echelon matrix

$r = 1 \qquad p = 1$

① $r \to r+1$; IF $r = m$ STOP

② IF $a_{rp} \neq 0$

FOR $i = r+1, \ldots, m$

replace row i by

$(\text{row } i) - \dfrac{a_{ip}}{a_{rp}} (\text{row } r)$

$r \to r+1$

①

eliminate entries below (r, p)

ELSE FOR $i = r+1, \ldots, m$

IF $a_{ip} \neq 0$

interch. row i and row p

②

interchange rows to have non zero entry at (r, p)

$p \to p+1$; ②

GAUSS ELIMINATION, BACK-SUBSTITUTION

INPUT: row echelon matrix $(a_{ij})_{1\le i\le m,\, 1\le j\le n}$

OUTPUT: reduced row echelon matrix

Start from the bottom row and move upward.
If the row only has zeros, move to the next row above.
Else, normalize so that the leading entry is 1, eliminate entries above the leading entry.

zero entry in row

EXAMPLE I

$$\begin{array}{cc} & \\ -2 \quad r & \\ & \\ & \\ & \end{array}\begin{array}{c}\left(\begin{array}{cccccc} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccccc} \quad & p' & \quad & \quad & \quad & \quad \end{array}\end{array}$$

4 x 6 row echelon
NOT reduced

r

$$\begin{array}{c}\left(\begin{array}{cccccc} 1 & 0 & 7 & 0 & 0 & -9 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccccc} p & \quad & \quad & \quad & \quad & \quad \end{array}\end{array}$$

$$\begin{cases} x_1 = -9 + 7x_3 \\ x_2 = 2 + 6x_3 + 3x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \\ x_5 = 0 \end{cases}$$

LECTURE 3: VECTOR EQUATIONS

① Vectors

DEF1: a vector is a matrix with only one column.

If it has d rows, we say it's a d-dimensional vector.

REMARKS: *) order matters: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

**) one can visualize a d-dimensional vector either as a point in $\mathbb{R}^d$ or as an arrow in $\mathbb{R}^d$. Those are not the only possible visualizations. ~~pos sible~~

PROBLEM 1: A) find the midpoint of the segment joining $(-1, 1)$ and $(2, 3)$.

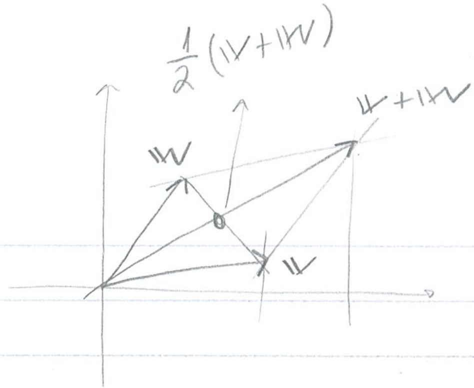
B) write coordinates for the vertices of a cube.

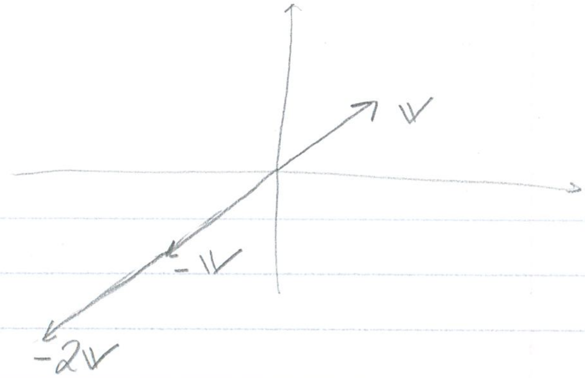
C) an equilateral triangle

② Vector algebra

DEF2: given $\mathbb{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}$, $\mathbb{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_d \end{bmatrix}$ and $\alpha \in \mathbb{R}$

define $\mathbb{v} + \mathbb{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_d + w_d \end{bmatrix}$ $\alpha \mathbb{v} = \begin{bmatrix} \alpha v_1 \\ \vdots \\ \alpha v_d \end{bmatrix}$.





DEF 3: Let $\mathbb{v}_1, \dots, \mathbb{v}_n \in \mathbb{R}^d$. A LINEAR COMBINATION of $\mathbb{v}_1, \dots, \mathbb{v}_n$ is any vector of the form

$\alpha_1 \mathbb{v}_1 + \dots + \alpha_n \mathbb{v}_n$ for some numbers $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

PROBLEM 2: draw all linear combinations of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

A) with integer coefficients

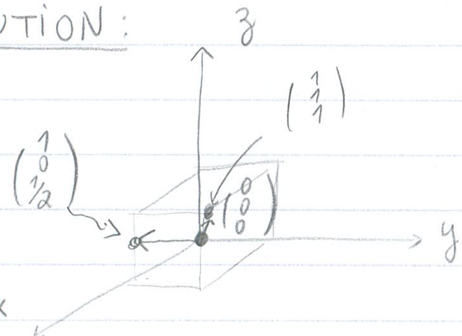
B) with positive coefficients

C) with rational coefficients

PROBLEM 3: Write down the equation of a plane that goes through $(0,0,0)^T$, $(1,1,1)^T$ and $(1,0,\frac{1}{2})^T$. I.e. find $F(x,y,z)$ such that

$$F(x,y,z) = 0 \iff (x,y,z) \in \text{plane}$$

SOLUTION:



the plane is the set of all linear combinations of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix}$, that is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix}$$

$$\begin{aligned} x &= a + b \\ y &= a \\ z &= a + \tfrac{1}{2} b \end{aligned}$$

write in terms of x, y and z only

$$\boxed{x + y - 2z = 0}$$

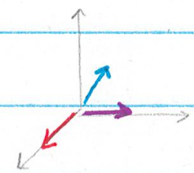
<u>REMARK</u>: the solutions to a 3 x 3 linear system

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$$

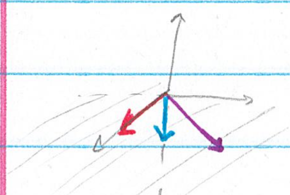
describe the intersection of three planes in space.

LECTURE 04: THE MATRIX EQUATION $Ax=b$

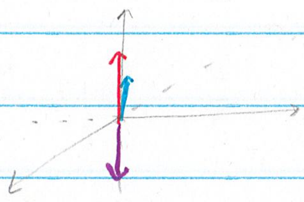
GOAL: given vectors $\vec{a_1}, \dots, \vec{a_n}$ all of the same dimension, find their span.



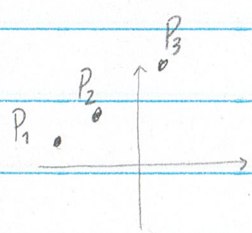
$\text{span} = \mathbb{R}^3$



$\text{span} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \ x_1 \text{ and } x_2 \text{ free variables} \right\}$



$\text{span} = \left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}, \ x_3 \text{ free variable} \right\}$



The points P_1, P_2, P_3 are aligned if $P_2 - P_1$ belongs to $\text{span}\{P_3 - P_1\}$

DEFINITION: Given a $m \times n$ matrix

$$A = [\vec{a_1} \; \vec{a_2} \; \cdots \; \vec{a_n}]$$

and a $m \times 1$ vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, define

$$A\vec{x} = x_1 \vec{a_1} + x_2 \vec{a_2} + \cdots + x_n \vec{a_n}.$$

EXAMPLES:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 28 \end{bmatrix} + \begin{bmatrix} 16 \\ 40 \end{bmatrix} + \begin{bmatrix} 27 \\ 54 \end{bmatrix}$$

$$= \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 7 [1] + 8 [2] + 9 [3]$$
$$= [50]$$

$$\begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = [122]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$

$$3\vec{a_1} + 5\vec{a_2} - 2\vec{a_3} = \begin{bmatrix} | & | & | \\ \vec{a_1} & \vec{a_2} & a_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$$

<u>DEFINITION</u>: given vectors $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix}$

define the DOT PRODUCT $\vec{v} \cdot \vec{w}$ as the number

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_d w_d .$$

<u>FACT</u>: If $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

then the segments OA and OB are perpendicular if and only if $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0.$

Back to the original problem.

QUESTION: does $\vec{b}$ belong to span $\{\vec{a_1}, \dots, \vec{a_n}\}$?

I.e. are there numbers $x_1, \dots, x_n$ such that

$x_1 \vec{a_1} + \dots + x_n \vec{a_n} = \vec{b}$?

Writing $A = [\vec{a_1} \cdots \vec{a_n}]$, the question can be rephrased as

"is there a vector $\vec{x}$ such that $A\vec{x} = \vec{b}$?"

$A\vec{x} = \vec{b}$ is a linear system of equations.

$$\left.\begin{aligned} x + 2y &= 3 \\ 2x + 3y &= 4 \end{aligned}\right| \text{ is the same as } \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

PROBLEMS

1) Is $\begin{bmatrix} 2 \\ -1 \\ 20 \end{bmatrix}$ a linear combination of

$\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -5 \\ 28 \end{bmatrix}$?

Yes if the system

$$\begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & -5 \\ 0 & 4 & 28 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 20 \end{bmatrix}$$

has a solution.

Do Gaussian elimination.

Eliminate entries below (1,1)

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & 7 & 5 \\ 0 & 4 & 28 & 20 \end{array}\right] \quad -4$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ reduced echelon

pivot positions

Solution: $\begin{cases} x = 2 - 4z \\ y = 5 - 7z \\ z \text{ is free} \end{cases}$

Answer: $\begin{bmatrix} 2 \\ -1 \\ 20 \end{bmatrix}$ can be written as a linear combination of $\begin{bmatrix} 2 \\ -1 \\ 20 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ in infinitely many ways. For example ($\cancel{z}\, z_3 = 0$)

$$\begin{bmatrix} 2 \\ -1 \\ 20 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

2) Do $\vec{v_1} = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} 0 \\ -6 \\ 12 \end{pmatrix}$ and $\vec{v_3} = \begin{pmatrix} 5 \\ -1 \\ 8 \end{pmatrix}$ span $\mathbb{R}^3$?

They do only if

$$\begin{pmatrix} 0 & 0 & 5 \\ 0 & -6 & -1 \\ -4 & 12 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$ has a solution for ANY $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Bring the coefficient matrix to row echelon see what happens to $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} -4 & 12 & 8 & c \\ 0 & -6 & -1 & b \\ 0 & 0 & 5 & a \end{array}\right) \begin{array}{l} \times -\frac{1}{4} \\ \times -\frac{1}{6} \\ \times \frac{1}{5} \end{array}$$

echelon form can see there is a solution

3) Ex. 14, page 41

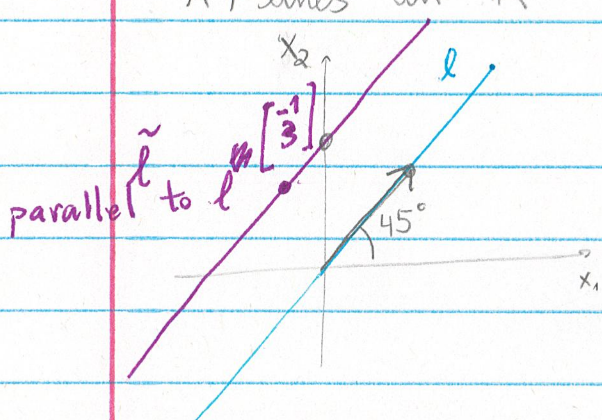
LECTURE 06: solution sets of linear systems

PROBLEM 1: a) given $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$, find the equation of the line that goes through $\vec{p}$ and $\vec{q}$.

b) given $\vec{p}$, $\vec{q}$ and $\vec{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$, find the equation of the plane that goes through $\vec{p}$, $\vec{q}$ and $\vec{r}$.

1) PARAMETRIC DESCRIPTIONS OF LINES AND PLANES

A) lines in $\mathbb{R}^2$



$\ell = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 \}$

ℓ is given by

$x_1 = x_2$ or $x_1 - x_2 = 0$ equation

ℓ is also given by

$c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, c free parametric vector form

$c \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, c free

$b \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, b free

What is the equation for $\tilde{\ell}$?

param. vector form: $\tilde{\ell} = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} + x \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x \text{ free} \right\}$

$= \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} + x \begin{bmatrix} -1 \\ -1 \end{bmatrix}, x \text{ free} \right\}$

equation:

$x_2 = x_1 + 4$

$= \left\{ \begin{bmatrix} 0 \\ 4 \end{bmatrix} + y \begin{bmatrix} 3 \\ 3 \end{bmatrix}, y \text{ free} \right\}$

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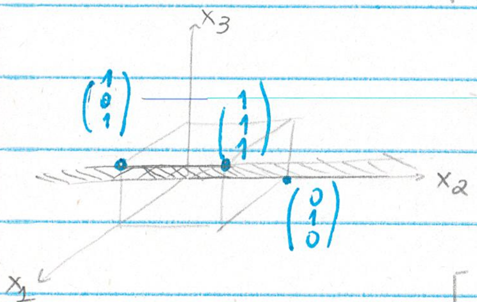
B) Planes in $\mathbb{R}^3$



equation

$x_1 = x_3$ or $x_1 - x_3 = 0$

parametric form (several poss.)

$$t\begin{bmatrix}1\\0\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\0\end{bmatrix}, \quad t \text{ and } s \text{ free}$$

Check that the parametric representation and the equation give the same set

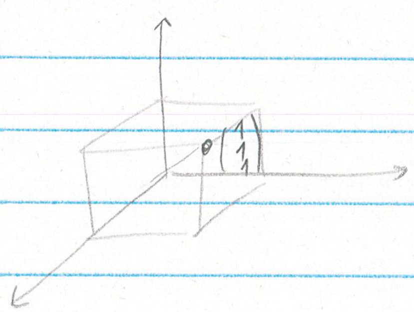
if $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}t\\s\\t\end{bmatrix}$ for some t and s

then $x_1 = x_3$

other representations:

$$s\begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}s+t\\s\\s+t\end{bmatrix}$$

C) Lines in $\mathbb{R}^3$

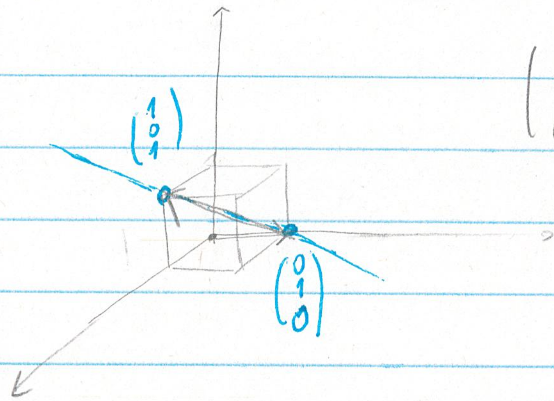


$x_1 = x_2 = x_3$

We need two equations

$$x_1 - x_2 = 0$$
$$x_2 - x_3 = 0$$

Parametric form: $t\begin{bmatrix}1\\1\\1\end{bmatrix}$, t free



$$\begin{pmatrix}0\\1\\0\end{pmatrix} + t\left\{\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} =$$

$$= \begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}t\\1-t\\t\end{pmatrix}$$

<u>2 equations:</u>

$$x_1 + x_2 = 1$$
$$x_2 + x_3 = 1$$

Solution to problem 1:

→ line through $\vec{p}$ and $\vec{q}$:

* if $\vec{p} \neq \vec{q}$

$$\vec{p} + t(\vec{q} - \vec{p}), \quad t \text{ free}$$

* if $\vec{p} = \vec{q}$

$$\vec{p} + t\vec{v}, \quad t \text{ free}, \ \vec{v} \text{ free}$$

~~Equations:~~

→ plane through $\vec{p}, \vec{q}$ and $\vec{r}$

$$\vec{p} + t(\vec{q} - \vec{p}) + s(\vec{r} - \vec{p}), \quad t, s \text{ free}$$

if $\vec{q} = \vec{p}$, replace $\vec{q} - \vec{p}$ by a free vector

if $\vec{r} = \vec{p}$, replace $\vec{r} - \vec{p}$ by a free vector (not the same)

Lecture 5: linear independence (Section 1.7)

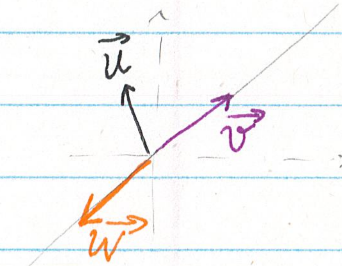
Recall $\text{span}\{\vec{a_1}, \dots, \vec{a_n}\}$ is the set of all linear combinations of $\vec{a_1}, \dots, \vec{a_n}$.

$\vec{b} \in \text{span}\{\vec{a_1}, \dots, \vec{a_n}\} \Longleftrightarrow \exists \vec{x}$ s.t. $A\vec{x} = \vec{b}$.

Examples in $\mathbb{R}^2$

$$\text{span}\{\vec{0}\} = \{\vec{0}\}$$

$$\text{span}\{\vec{0}, \vec{v}\} = \{c\vec{v} : c \in \mathbb{R}\}$$



$$\text{span}\{\vec{v}, \vec{w}\} = \text{span}\{\vec{v}\}$$
$$= \text{span}\{\vec{w}\}$$

$$\text{span}\{\vec{u}, \vec{v}\} = \mathbb{R}^2$$
$$= \text{span}\{\vec{u}, \vec{w}\}$$
$$= \text{span}\{\vec{u}, \vec{v}, \vec{w}\}$$

Examples in $\mathbb{R}^4$

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\} = \left\{\begin{bmatrix}x_1\\x_2\\0\\0\end{bmatrix} : x_1, x_2 \text{ free}\right\}$$

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right\} = \mathbb{R}^4$$

$$= \text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

02

$$\text{span}\left\{\underset{\vec{a_1}}{\begin{bmatrix}-4\\0\\1\\5\end{bmatrix}}, \underset{\vec{a_2}}{\begin{bmatrix}-3\\-1\\0\\4\end{bmatrix}}, \underset{\vec{a_3}}{\begin{bmatrix}0\\4\\3\\6\end{bmatrix}}\right\} = ?$$

$\vec{b} \in \text{span}\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ if and only if

there exists $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ s.t. $A\vec{x} = \vec{b}$.

$$\left[\begin{array}{ccc|c} -4 & -3 & 0 & b_1 \\ 0 & -1 & 4 & b_2 \\ 1 & 0 & 3 & b_3 \\ 5 & 4 & 6 & b_4 \end{array}\right]$$

find reduced row echelon form

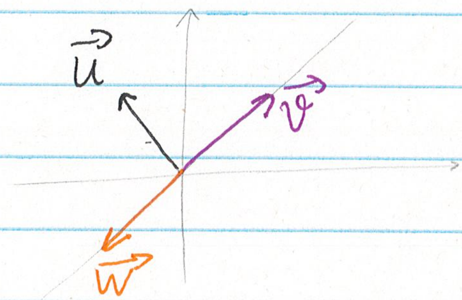
Each $[0 \ 0 \ 0 \mid ?]$ row gives a linear equation that the b_i must satisfy.

DEFINITION: the vectors $\{\vec{v}_1, \dots, \vec{v}_n\} \in \mathbb{R}^m$ are LINEARLY INDEPENDENT (LI) if the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n \quad (\star)$$

has ONLY ONE solution, $x_1 = x_2 = \dots = x_n = 0$.

They are LINEARLY DEPENDENT if $(\star)$ has a solution with not all x_i equal to zero.



$\{\vec{v}\}$	LI
$\{\vec{u}, \vec{v}\}$	LI
$\{\vec{v}, \vec{w}\}$	LD
$\{\vec{u}, \vec{v}, \vec{w}\}$	LD
$\{\vec{u}, \vec{v}, \vec{0}\}$	LD

FACT: if a set of vectors is LD, one of them can be removed without changing their span.

PROBLEM: Describe the possible row echelon forms of a 4×3 matrix and decide if the columns are LI or LD.

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix} \quad \text{LI} \qquad\qquad \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{LD}$$

$\blacksquare$ nonzero entry

$*$ anything

$$\begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{LD} \qquad \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{LD}$$

$$\begin{bmatrix} 0 & 0 & \blacksquare \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{LD} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{LD}$$

PROBLEM: are the columns of A LI or LD?

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix} \; 3$$

SOLUTION: LI if the ONLY solution of $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

Row reduce $[A \;\; 0]$

$$\begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \text{LI}$$

REMARK: if a matrix has more columns than rows, its columns are LD.

If a linear system has more variables than equations, at least one of the variables is free.

LECTURE 07: LINEAR TRANSFORMATIONS I

Recall that the product $A\vec{x}$ of

$$A = [\,\vec{a_1} \cdots \vec{a_n}\,] \quad \text{and} \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is the vector $A\vec{x} = x_1\vec{a_1} + \cdots + x_n\vec{a_n}$.

If we regard A as given and $\vec{x}$ as variable, then the rule $\vec{x} \mapsto A\vec{x}$ defines a FUNCTION with domain $\mathbb{R}^m$ (m = # rows in A) and codomain $\mathbb{R}^n$.

Review of functions

DOMAIN	CODOMAIN	RULE
$\mathbb{R}$	$\mathbb{R}$	$x \mapsto \cos x$
$\mathbb{R}$	$\mathbb{R}$	$x \mapsto x^2$
$\mathbb{N}$	$\mathbb{N}$	$n \mapsto n+1$
$\mathbb{R}$	$\mathbb{R}^2$	$t \mapsto t\begin{bmatrix}1\\2\end{bmatrix}$
$\mathbb{R}$	$\mathbb{R}^2$	$t \mapsto t\begin{bmatrix}1\\2\end{bmatrix} + (1-t)\begin{bmatrix}-1\\1\end{bmatrix}$
$\mathbb{R}^2$	$\mathbb{R}^2$	rotates $90°$ counterclockwise
$\mathbb{R}^3$	$\mathbb{R}^3$	project onto horiz. plane through the origin
$\mathbb{R}^n$	$\mathbb{R}^m$	$\vec{x} \mapsto A\vec{x}$, A is an $m\times n$ matrix

DEF: A function $A : \mathbb{R}^n \to \mathbb{R}^m$ is LINEAR if

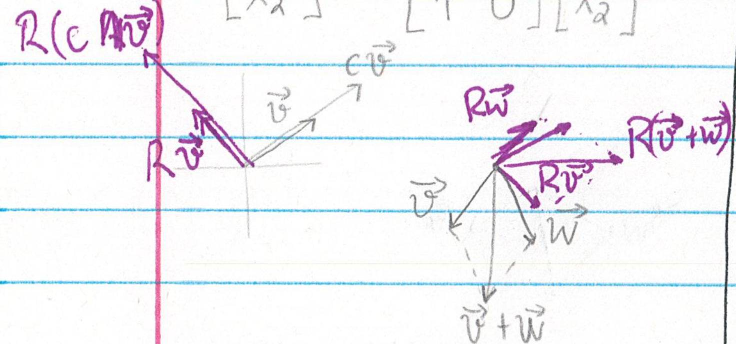
$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} \qquad \text{for any } \vec{x}, \vec{y} \text{ in } \mathbb{R}^n$$

$$A(c\vec{x}) = c \cdot A\vec{x} \qquad \text{for any } \vec{x} \text{ in } \mathbb{R}^n \text{ and } c \text{ in } \mathbb{R}$$

EXAMPLES

LINEAR	NOT LINEAR
$\mathbb{R} \qquad \mathbb{R}$ $x \longmapsto 2x$	$\mathbb{R} \qquad \mathbb{R}$ $x \longmapsto 2x + 1$
$\mathbb{R}^3 \qquad \mathbb{R}^3$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	$\mathbb{R}^3 \qquad \mathbb{R}^3$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
$\mathbb{R}^3 \qquad \mathbb{R}^3$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	$\mathbb{R}^3 \qquad \mathbb{R}^3$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ 2x_2 + \frac{1}{2} \\ 3x_3 \end{bmatrix}$

LINEAR	NOT LINEAR
$\mathbb{R}^2 \xrightarrow{R} \mathbb{R}^2$	$\mathbb{R}^2 \xrightarrow{\tilde{R}} \mathbb{R}^2$
$\vec{x} \mapsto$ rotate $\vec{x}$ 90° counterclockwise around $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\vec{x} \mapsto$ rotate 90° counter-clockwise around $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\tilde{R}\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \neq 2 \cdot \tilde{R}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



SOME PROPERTIES: if $A: \mathbb{R}^n \to \mathbb{R}^m$ is a linear function then

*) $A(\vec{0}) = \vec{0}$ (because $A(\vec{0}) = A(\vec{0}-\vec{0}) = A(\vec{0}) - A(\vec{0})$)

**) A maps LD sets into LD sets

***) A linear map $B: \mathbb{R}^3 \to \mathbb{R}^3$ maps collinear points into collinear points, and coplanar poits into coplanar points.

Lecture 09: matrix operations

Recall: given a $m \times n$ matrix $A = [\vec{a_1} \cdots \vec{a_n}]$ and a vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, the product $A\vec{x}$ is the $m \times 1$ vector

$$A\vec{x} = x_1 \vec{a_1} + \cdots + x_n \vec{a_n}$$

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ 6 \\ 15 \end{bmatrix}$$

DEFINITION: if A is a $m \times n$ matrix and $B = [\vec{b_1} \ \vec{b_2} \cdots \vec{b_p}]$ is an $n \times p$ matrix then AB is defined as the $m \times p$ matrix

$$AB = [A\vec{b_1} \ A\vec{b_2} \cdots A\vec{b_p}]$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix}}_{3\times 3 \text{ matrix}} \overbrace{\begin{bmatrix} 2 & 1 \\ 5 & 2 \\ 0 & 3 \end{bmatrix}}^{3\times 2 \text{ matrix}} = \underbrace{\begin{bmatrix} 7 & 21 \\ 6 & 6 \\ 7 & 15 \end{bmatrix}}_{3\times 2 \text{ matrix}}$$

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{2\times 2} = \overset{2\times 2}{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1\times 3} \underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{3\times 1} = \begin{bmatrix} 1\cdot 4 + 2\cdot 5 + 3\cdot 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{3\times 1} \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1\times 3} = \underbrace{\begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}}_{3\times 3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

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Skip

DEFINITION: if A is an $m \times n$ matrix, the TRANSPOSE of A is the $n \times m$ matrix A^T defined by

$$(\text{entry } (i,j) \text{ of } A^T) = (\text{entry } (j,i) \text{ of } A)$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

the columns of A^T are the rows of A

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

The DOT PRODUCT of the vectors $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ is the only entry of $\vec{v}^T \vec{w}$.

ALTERNATIVE DEFINITIONS OF PRODUCT

A $m \times n$ matrix

B $n \times p$ matrix

AB $m \times p$ matrix

BY ENTRIES

entry (i,j) of AB = (row i of A)(column j of B)

BY ROWS

row i of AB = (row i of A)

BY COLUMNS

column j of AB = A(column j of B)

04

by rows or by columns

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$$

by rows

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} a_1 & 2a_2 & 3a_3 \\ b_1 & 2b_2 & 3b_3 \\ c_1 & 2c_2 & 3c_3 \end{bmatrix}$$

by columns

$$\begin{bmatrix} & ? & \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c+a & d+b \end{bmatrix}$$

by rows, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} & ? & \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -8 \end{bmatrix}$$

by rows, $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$

05

REMARK: here are some rules of algebra that are FALSE

*) $AB = BA$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{projection on the } x_1\text{-axis}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{90^\circ \circlearrowleft \text{ rotation}} \neq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

**) $AB = AC \not\Rightarrow B = C$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

EXTRA EXAMPLE

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\sin\alpha\cos\beta + \sin\beta\cos\alpha) \\ \sin\alpha\cos\beta + \sin\beta\cos\alpha & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

Lecture 12: subspaces and bases (Section 2.8)

DEFINITION: let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in $\mathbb{R}^m$ and
$V = [\vec{v}_1 \cdots \vec{v}_n]$.
We say that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is LINEARLY INDEPENDENT if the ~~equal~~ linear system

$$V\vec{x} = \vec{0}$$

has ONLY ONE solution, $\vec{x} = \vec{0}$.

EXAMPLE 1: ~~x~~ the columns of the identity matrix are LI.

EXAMPLE 2 (IMPORTANT): the pivot colums of a row echelon matrix are LI

$$V = \begin{bmatrix} 1 & 4 & 5 & 7 \\ 0 & 1 & 2 & 32 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DEFINITION: a SUBSPACE of $\mathbb{R}^n$ is a set of vectors closed under addition and multiplication by numbers.

EXAMPLES: if $A = [\vec{a_1} \cdots \vec{a_n}]$ is an $m \times n$ matrix, we can define

•) The COLUMN SPACE of A is

$$\text{Col } A := \text{span}\{\vec{a_1}, \cdots, \vec{a_n}\}$$

is a subspace of $\mathbb{R}^m$.

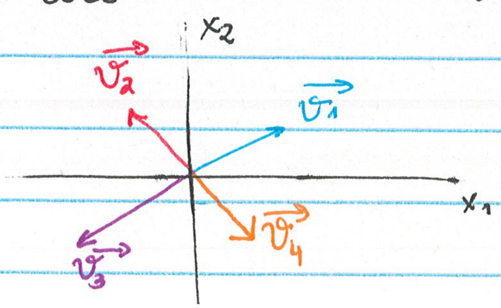
••) The NULL SPACE of A is

$$\text{Nul } A := \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \underbrace{\vec{0}}_{\text{in } \mathbb{R}^m}\}$$

EXERCISE: check that Col A and Nul A are subspaces.

DEFINITION: a BASIS of a subspace of $\mathbb{R}^n$ is a linearly independent set that spans the space.

EXERCISE: ~~among the~~ find all subsets of $\{v_1, v_2, v_3, v_4\}$, ~~find all~~ which are bases of $\mathbb{R}^2$.



$\{\vec{v_1}, \vec{v_2}\}$ $\{\vec{v_3}, \vec{v_4}\}$

$\{\vec{v_1}, \vec{v_4}\}$

$\{\vec{v_2}, \vec{v_3}\}$

EXERCISE: find a basis for the column space of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 7 & 2 & 8 \\ 0 & 0 & 1 & 3 & 4 & 23 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Pivot columns $1^{st}, 3^{rd}, 4^{th}$ and 6^{th} column are LI. Do they generate the column space? Yes, because

2^{nd} column is in the span of the 1^{st}

5^{th} column is in the span of the $1^{st}, 3^{rd}$ and 4^{th}.

EXERCISE: find a basis for the column space of the matrix (compare this problem with the problem of Lecture 4)

$$\begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$$

Call the matrix A and its columns $\vec{a_1}, \dots, \vec{a_4}$.

Row reduce to

$$B = \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\{\vec{b_1}, \vec{b_2}\}$ is LI, $\{\vec{a_1}, \vec{a_2}\}$ is LI (why?)

Since $\{\vec{b_3}, \vec{b_4}\} \subset \operatorname{span}\{\vec{b_1}, \vec{b_2}\}$, also

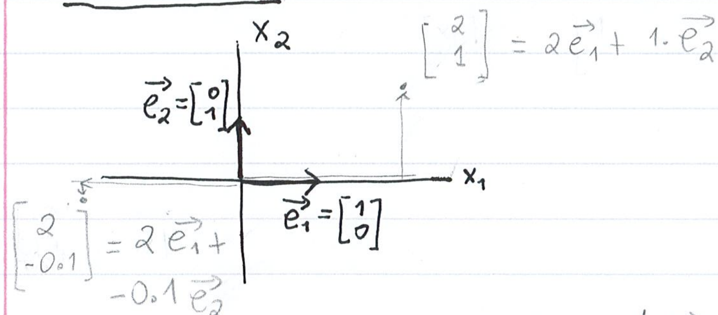
$\{\vec{a_3}, \vec{a_4}\} \subset \operatorname{span}\{\vec{a_1}, \vec{a_2}\}$ (why?)

Lecture 13: dimension and rank

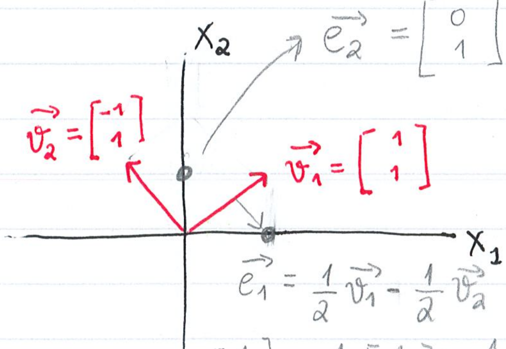
DEFINITION: a SUBSPACE of $\mathbb{R}^n$ is a set of the form $\text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$ for some choice of $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$.

A BASIS of a subspace is a linearly independent set that spans the subspace.

EXAMPLE: bases in $\mathbb{R}^2$

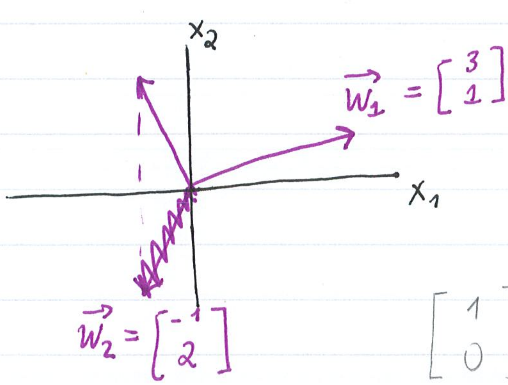


$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$



$\vec{e}_1 = \frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$



PROBLEM: write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the basis $\{\vec{w}_1, \vec{w}_2\}$.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = y_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

02

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

$$-3 \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 0 & -7 & 1 & -3 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{7} & \frac{3}{7} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & \frac{1}{7} \\ 0 & 1 & -\frac{1}{7} & \frac{3}{7} \end{array}\right]$$

<u>Answer</u>: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{7}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

<u>THEOREM</u>: all the bases of a subspace have the same number of elements. This number is called the DIMENSION of the subspace.

do problem on page 4

PROBLEM: find bases for col A and nul A, where

$$A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

•) The pivot columns form a basis for col A.

••) Each nonpivot column gives a free variable in $A\vec{x} = 0$.

$$A \sim \begin{bmatrix} 1 & -3 & 0 & -\frac{6}{5} \\ 0 & 0 & 1 & -\frac{7}{5} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution of $A\vec{x} = 0$.

$$\begin{cases} x_1 = 3x_2 + \frac{6}{5}x_4 \\ x_2 \text{ free} \\ x_3 = \frac{7}{5}x_4 \\ x_4 \text{ free} \end{cases}$$

Basis for nul A: $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{6}{5} \\ 0 \\ \frac{7}{5} \\ 1 \end{bmatrix}$.

~~THEOREM:~~

DEFINITION: the RANK of a matrix A is the dimension of the column space of A.

04

PROBLEM: determine the dimension of the subspace of $\mathbb{R}^3$ spanned by

$$\begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 6 \\ -7 \end{bmatrix}.$$

The dimension is the rank of

$$\begin{bmatrix} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -10 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & -6 \end{bmatrix}$$

There are 3 pivots, so the rank is 3.

(skip) PROBLEM: find the coordinates of the vector $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$ in the basis $\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$.

We look for numbers x_1 and x_2 such that

$$\begin{bmatrix} -7 \\ 5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 5 \end{bmatrix},$$

that is

$$\begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

Row reduce

$$\left[\begin{array}{cc|c} 1 & -3 & -7 \\ 0 & -4 & -16 \end{array}\right] \qquad \begin{aligned} x_2 &= 4 \\ x_1 &= 5 \end{aligned}$$

THEOREM: Let A be an $m \times n$ matrix. Then

$$\boxed{\operatorname{rank} A + \dim \operatorname{Nul} A = n}$$

EXAMPLE: $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$\operatorname{rank} A = 1$

$\operatorname{Nul} A = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$. $\quad 1 + 2 = 3$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \operatorname{rank} A = 2$

$\operatorname{Nul} A = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\} \quad 2 + 1 = 3$

Lecture 1~~3~~4: Determinants

DEFINITION: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the DETERMINANT of A is defined as the number

~~det A = ab =~~ $\det A = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

PROBLEM: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Solve $A\vec{x} = \begin{bmatrix} u \\ v \end{bmatrix}$ and compute the inverse A^{-1} if it exists.

$$\begin{array}{l} \times c \\ \times a \end{array} \begin{cases} a x_1 + b x_2 = u \\ c x_1 + d x_2 = v \end{cases} \qquad \begin{cases} ac\, x_1 + bc\, x_2 = uc \\ ac\, x_1 + ad\, x_2 = av \end{cases}$$

Let $D = \det A$. Then

$$\begin{cases} ac\, x_1 + bc\, x_2 = uc \\ \qquad\quad D\, x_2 = av - uc = \begin{vmatrix} a & u \\ c & v \end{vmatrix} \end{cases}$$

If $D \neq 0$ then

$$x_2 = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Repeating the argument,

$$x_1 = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

If $D = 0$ then there are choices of $\begin{bmatrix} u \\ v \end{bmatrix}$ for which the system is inconsistent.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is NOT invertible

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible.

1st column of A^{-1} = solution of $A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} \begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix} \\ \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} \end{bmatrix} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d \\ -c \end{bmatrix}$$

2nd column of A^{-1} = solution of $A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} \begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix} \\ \begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix} \end{bmatrix} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} -b \\ a \end{bmatrix}$$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Check!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

How to remember the formula for the inverse matrix A^{-1}

→ in each row and column, the signs change

→ the entry (i,j) is, except for a $+$ or $-$, the entry that remains after removing row i and column j of A.

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do it by columns instead of rows

DEFINITION: Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. The DETERMINANT is defined as

$$\det A = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

THM (CRAMER'S RULE) ~~There~~ If $\det A \neq 0$, there is a formula for the solution of $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$:

$$x_1 = \frac{1}{\det A} \begin{vmatrix} u & a_2 & a_3 \\ v & b_2 & b_3 \\ w & c_2 & c_3 \end{vmatrix}$$

$$x_2 = \frac{1}{\det A} \begin{vmatrix} a_1 & u & a_3 \\ b_1 & v & b_3 \\ c_1 & w & c_3 \end{vmatrix}$$

$$x_3 = \frac{1}{\det A} \begin{vmatrix} a_1 & a_2 & u \\ b_1 & b_2 & v \\ c_1 & c_2 & w \end{vmatrix}$$

~~Lecture 12: subspaces of $\mathbb{R}^n$~~

EXERCISE: Compute $\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = M$.

By definition,

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

$$= -2$$

What happens if you use the second row instead of the first?

$$2 \begin{vmatrix} 5 & 0 \\ -2 & 0 \end{vmatrix} - 4 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = 2$$

$$= -\det M$$

2nd column: $5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} - 4 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = 2$

3rd column: $0 \begin{vmatrix} & \\ & \end{vmatrix} + 1 \begin{vmatrix} & \\ & \end{vmatrix} + 0 \begin{vmatrix} & \\ & \end{vmatrix} = -2$

DEFINITION: Let A be a square matrix $n \times n$.

Let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j.

The (i,j)-COFACTOR is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

The DETERMINANT of A is ~~defined as~~ the number

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

THM: given any row i and column j, it holds

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}.$$

EXERCISE: compute $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix}$

Lecture 15: determinants II

PROPERTIES OF DETERMINANTS

① The determinant of a ~~triangular~~ echelon matrix is the product of its diagonal elements.

1st column

$$\begin{vmatrix} a & x & y \\ 0 & b & w \\ 0 & 0 & c \end{vmatrix} = a \begin{vmatrix} b & w \\ 0 & c \end{vmatrix} - 0 \begin{vmatrix} x & y \\ 0 & c \end{vmatrix} + 0 \begin{vmatrix} x & y \\ b & w \end{vmatrix}$$

$$= abc$$

~~1st row~~ 3x3 echelon

$$\begin{vmatrix} a & 0 & 0 & 0 \\ x & b & 0 & 0 \\ y & z & c & 0 \\ w & u & v & d \end{vmatrix} = a \begin{vmatrix} b & 0 & 0 \\ z & c & 0 \\ u & v & d \end{vmatrix} = abcd$$

② ~~The~~ When two rows (or columns) are interchanged, the determinant changes sign.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

③ The determinant is linear in each row and column separately.

~~In other words: if I replace (row i) by (row i) + c(row j), the determinant does not change.~~

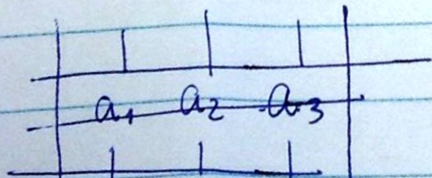
$$\begin{vmatrix} a+\tilde{a} & b+\tilde{b} \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \tilde{a} & \tilde{b} \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a_1+\tilde{a}_1 & b_1 & c_1 \\ a_2+\tilde{a}_2 & b_2 & c_2 \\ a_3+\tilde{a}_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \tilde{a}_1 & b_1 & c_1 \\ \tilde{a}_2 & b_2 & c_2 \\ \tilde{a}_3 & b_3 & c_3 \end{vmatrix}$$

Because

$$LHS = (a_1+\tilde{a}_1)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (a_2+\tilde{a}_2)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (a_3+\tilde{a}_3)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

~~PROPERTY ③ IN SYMBOLS~~



CONSEQUENCES OF PROPERTY ③: if a row operation is performed, the determinant does not change, UNLESS the operation is row exchange.

$$\begin{vmatrix} - & \vec{a}_1 & - \\ - & \vec{a}_2 & - \\ - & \vec{a}_3 + c\vec{a}_1 & - \end{vmatrix} \overset{③}{=} \begin{vmatrix} - & a_1 & - \\ - & a_2 & - \\ - & a_3 & - \end{vmatrix} + c \begin{vmatrix} - & a_1 & - \\ - & a_2 & - \\ - & a_3 & - \end{vmatrix}$$

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(5) EXERCISE: compute

$$\begin{vmatrix} 1 & 5 & -7 \\ -1 & -4 & -5 \\ 2 & 8 & 7 \end{vmatrix}$$

The determinant does not change if we replace (row 2) by (row 2)+(row 1). In other words, we can row reduce the matrix without changing its determinant.

$$\begin{vmatrix} 1 & 5 & -7 \\ 0 & 1 & -12 \\ 0 & -2 & 21 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -7 \\ 0 & 1 & -12 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= 1 \cdot 1 \cdot (-3) = -3$$

(1) EXERCISE: compute

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 0$$

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(4) **EXERCISE:** find a formula for

$\det(rA)$ where A is a $n \times n$ matrix and r is a number.

(2) **PROBLEM:** is it true that $\det(A+B) = \det A + \det B$?

No! $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \neq \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

(3) **EXERCISE:** compute

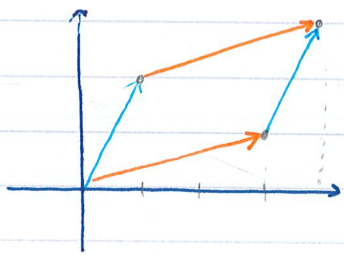
$$\begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 0 & 0 & 2 & 1 \end{vmatrix}$$

$$= -4 \begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 3 \\ 0 & 2 & 1 \end{vmatrix} = -4 \begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & -3 \\ 0 & 2 & 1 \end{vmatrix}$$

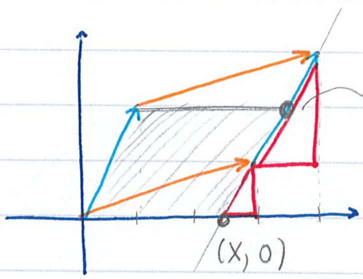
$$= -4 \begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & -3 \\ 0 & 0 & -2 \end{vmatrix} = -4 \cdot 3(-2) \cdot (-2) = -48.$$

Lecture 16: determinants III

PROBLEM: compute the area of the parallelogram ~~wih~~ with vertices $(0,0)$, $(3,1)$, $(1,2)$ and $(4,3)$



We could do base x heigth, but what is the heigth?



what are the coordinates here?

$(X,0)+(1,2) = (X+1,2)$

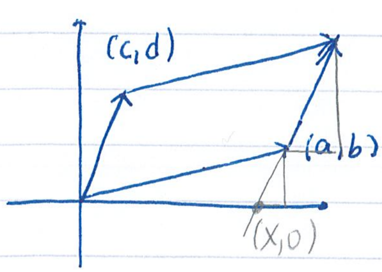
What is x? Look at the red triangles.

$$\frac{1}{3-X} = \frac{3}{1} \Rightarrow X = \frac{8}{3}$$

$$\text{Area} = (\text{base} \times \text{heigth}) = \frac{8}{3}\cdot 2 = \frac{16}{3}$$

PROBLEM 2: replace $(3,1)$ and $(1,2)$ by (a,b) and (c,d).

Assume $a > c > 0$, $d > b > 0$.



$$\frac{d}{c} = \frac{b}{a-x} \qquad \text{Area} = xd$$

$$ad - xd = bc \Rightarrow xd = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

THEOREM: Let $\vec{v_1}, \dots, \vec{v_n}$ in $\mathbb{R}^n$. The PARALLELEPIPED spanned by $(\vec{v_1}, \dots, \vec{v_n})$ is the set of linear combinations of $\vec{v_1}, \dots, \vec{v_n}$ with coefficients in $[0, 1]$.

The n-dimensional VOLUME of the parallelepiped spanned by $\vec{v_1}, \dots, \vec{v_n}$ is $\left|\det\left[\vec{v_1} \cdots \vec{v_n}\right]\right|$.

REMARKS: A) In 3 dimensions, ~~the area is b~~

volume = (area of base) × (heigth)

$$\left|\det\begin{bmatrix} h & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}\right| = |h| \cdot |ad - bc|$$

B) ~~If~~ The columns of a square matrix are LD if and only if the parallelepiped they span has volume zero.

C) The linear map whose matrix is $V = \left[\vec{v_1} \cdots \vec{v_n}\right]$ stretches volumes by a factor $|\det V|$.

C.1) $$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = (\cos\theta)^2 + (\sin\theta)^2 = 1$$

C 2) $$\begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} = 12$$

D) If ~~one of~~ $\vec{v_j}$ is replaced by $c\vec{v_j}$ then the volume of the parallelepiped is multiplied by $|c|$.

<u>THEOREM:</u> $\det(AB) = \det A \cdot \det B$

$$\det(A^T) = \det A$$

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LECTURE 18: LINEAR TRANSFORMATIONS (SECTION 4.2)

EXAMPLES OF VECTOR SPACES

show the zeros

A) $\mathbb{P}_n$ polynomials of degree at most n

B) $m \times n$ matrices

C) null space of a matrix

DEFINITION: ~~A LINEAR MAP is a function~~

A LINEAR TRANSFORMATION from a vector space V to a vector space W is a rule that assigns to each x in V a unique T(x) in W, such that

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \text{for all } \vec{u}, \vec{v} \text{ in } V$$
$$T(c\vec{u}) = cT(\vec{u}) \quad \text{for all } \vec{u} \text{ in } V \text{ and numbers } c.$$

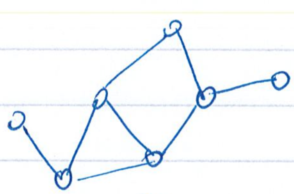
EXAMPLES: A) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B) $\frac{d}{dt}: \mathbb{P}_n \longrightarrow \mathbb{P}_{n-1}$

$$p(t) \longmapsto p'(t)$$

(skip) C)



X

$$\Delta: \mathbb{R}^X \longrightarrow \mathbb{R}^X$$

$$\Delta f(x) := \frac{1}{d(x)} \sum_{y \sim x} [f(y)]$$

$d(x)$ = degree of x = number of neighbors of x

Δ replaces each number with the average of its neighbors

<u>PROBLEM 1</u>: show that if $T: V \to W$ is a linear transformation then $T(0) = 0$.

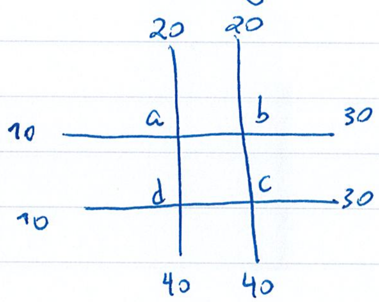
$$\underset{0 \text{ in } V}{T(0)} = \underset{0 \text{ in } W}{0}.$$

$$T(0) = T(0+0) = T(0) + T(0) \Rightarrow T(0) = 0$$

or

$$\underbrace{T(\vec{0}_V)}_{\text{zero vector in } V} = T(\underbrace{0}_{\text{number zero}} \cdot \underbrace{\vec{0}_V}_{\text{zero vector}}) = 0 \cdot T(\vec{0}_V) = \vec{0}_W.$$

<u>PROBLEM 2</u>: find numbers a, b, c and such that each number is the average of its neighbors

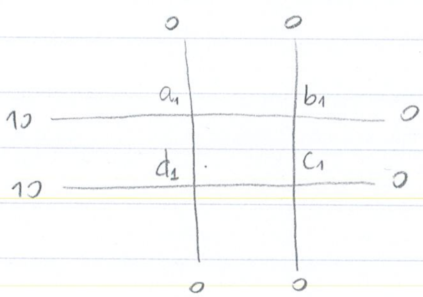


$$a = \frac{10 + 20 + b + d}{4}$$

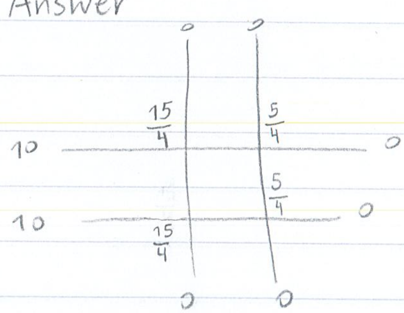
$$b = \frac{a + 20 + 30 + c}{4}$$

$$c = \frac{d + b + 30 + 40}{4}$$

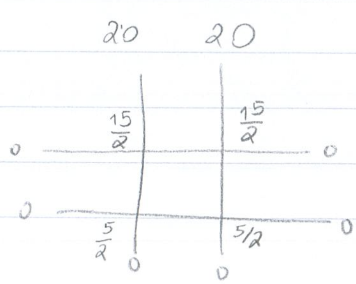
Start with simplified version:

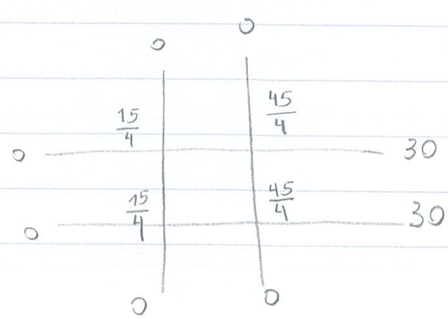


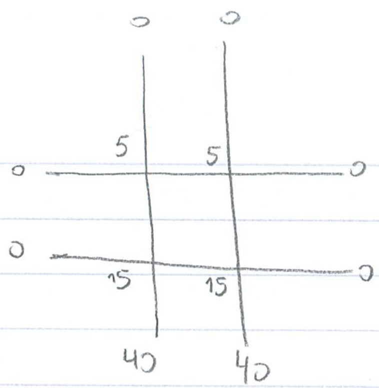
Answer



From this, infer







We can add those to get

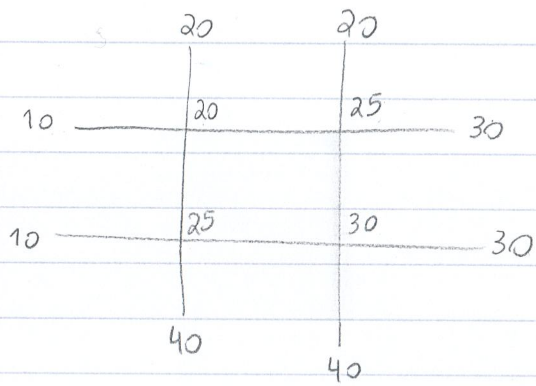
$$a = \frac{15}{4} + \frac{15}{2} + \frac{15}{4} + 5 = 20$$

$$b = \frac{5}{4} + \frac{15}{2} + \frac{45}{4} + 5 = 25$$

$$c = \frac{5}{4} + \frac{5}{2} + \frac{45}{4} + 15 = 30$$

$$d = \frac{15}{4} + \frac{5}{2} + \frac{15}{4} + 15 = 25$$

Check:



What is the relationship with linear transformations?

We were looking for a, b, c, d such that

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \underbrace{\frac{1}{4}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{x} + \underbrace{\frac{1}{4}\begin{bmatrix} 10+20 \\ 20+30 \\ 30+40 \\ 40+10 \end{bmatrix}}_{b}$$

that is, $x = Ax + b$. We have split b into a sum

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$$b = \underbrace{\frac{1}{4}\begin{bmatrix}10\\0\\0\\10\end{bmatrix}}_{b_1} + \underbrace{\frac{1}{4}\begin{bmatrix}20\\20\\0\\0\end{bmatrix}}_{b_2} + \underbrace{\frac{1}{4}\begin{bmatrix}0\\30\\30\\0\end{bmatrix}}_{b_3} + \underbrace{\frac{1}{4}\begin{bmatrix}0\\0\\40\\40\end{bmatrix}}_{b_4}$$

and solved for

$$\left.\begin{aligned} x_1 &= Ax_1 + b_1 \\ x_2 &= Ax_2 + b_2 \\ x_3 &= Ax_3 + b_3 \\ x_4 &= Ax_4 + b_4 \end{aligned}\right| \Rightarrow (x_1+x_2+x_3+x_4) = A(x_1+x_2+x_3+x_4) + (b_1+b_2+b_3+b_4)$$

<u>DEFINITION</u>: Let $T: V \to W$ be a linear transformation between the spaces V and W.

The KERNEL of T is the subset of V

$N(T)$ ~~Ker T~~ $= \{\vec{x} \in V : T\vec{x} = \vec{0}\}$

The RANGE of T is the ~~set~~ subset of W

~~Ran T =~~ $R(T) = \{\vec{y} \in W : \vec{y} = T\vec{x}$ for some $\vec{x}$ in $V\}$

<u>REMARK</u>: $N(T)$ is a subspace of V

$R(T)$ is a subspace of W.

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$ for a $m \times n$ matrix A, then $N(T) = \text{Nul } A$ and $R(T) = \text{Col } A$.

LECTURE 19: LINEAR INDEPENDENCE AND BASES (SECTION 4.3)

DEFINITION: Let V be a vector space and $v_1, \dots, v_n$ be elements of V. We say that the set $\{v_1, \dots, v_n\}$ is LINEAR INDEPENDENT if the equation $x_1 v_1 + \dots + x_n v_n = 0$ $x_1, \dots, x_n$ in $\mathbb{R}$ is true ONLY for $x_1 = \dots = x_n = 0$.

The set $\{v_1, \dots, v_n\}$ is LINEAR DEPENDENT if it is not linear independent.

EXAMPLES:

VECTOR SPACE	L I	L D
$\mathbb{R}^3$	$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ columns If $x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = 0$ then $x_1 + 2x_2 + 3x_3 = 0$ $x_2 + 4x_3 = 0$ $x_3 = 0$ so $x_1 = x_2 = x_3 = 0$. Therefore the columns of A are LI. What about the rows?	$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ The columns are LD because $5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

In $\mathbb{P}_3$ (polynomials of degree at most 3),

$\{1, t, t^2, t^3\}$ is LI, because the only choice of a, b, c, d that make $a + bt + ct^3 + dt^4 = 0$ for all t is the choice $a = b = c = d = 0$.

$\{t+1, t+2, t^2, 1\}$ is LD because

$(t+1) - (t+2) + 0 \cdot t^2 + 1 = 0$.

<u>DEFINITION</u>: A BASIS for a vector space V is a LI set that spans V.

<u>EXAMPLES</u>: A) the columns of ~~an invertible matrix~~ the identity matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ form a basis for $\mathbb{R}^4$.

B) $\{1, t, t^2, t^3, t^4\}$ is a basis for $\mathbb{P}_4$

<u>PROBLEM 1</u>: Find a basis for the plane

$$x + 2y + 3z = 0.$$

Can take $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. They are clearly LI. Why do they span the plane?

In other words: given x, y, z such that

$$x + 2y + 3z = 0,$$

find a and b such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

If we choose $a = y$ and $b = z$ then

$$-2a - 3b = -2y - 3z \underset{\substack{\uparrow \\ \text{because } x+2y+3z=0}}{=} x$$

Can you complete that to a basis of $\mathbb{R}^3$?

PROBLEM 2: In the vector space of all functions $f: \mathbb{R} \to \mathbb{R}$,

find a basis for the subspace spanned by

~~{sin t, cos t,~~ $\{\sin t, \sin 2t, \sin t \cos t\}$.

Remember $\sin 2t = 2 \sin t \cos t$. Can remove $\sin 2t$

from the set without affecting their span. Is

$\{\sin t, \sin t \cos t\}$ LI? Yes, because $\sin t$ and $\cos t$

are not multiples.

PROBLEM 3: Is the ~~set~~ subset of $\mathbb{P}_4$

$$\{t^3-5t^2+1,\ 2t^4+5t-6,\ t^2-5t+2\} \quad \text{LI or LD} \quad ?$$

Going back to the definition: is there $(x_1, x_2, x_3) \neq (0,0,0)$

such that

$$x_1(t^3-5t^2+1)+x_2(2t^4+5t-6)+x_3(t^2-5t+2)=0 \quad ?$$

$$2x_2t^4+x_1t^3+(-5x_1+x_3)t^2+(5x_2-5x_3)t+(x_1-6x_2+2x_3)=0$$

$\Rightarrow x_1=x_2=x_3=0 \Rightarrow$ the set is LI.

PROBLEM 4: Is the set of 2×2 matrices

$$\begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix} \quad \text{LI or LD} \quad ?$$

We look for x_1, x_2, x_3 such that

$$x_1\begin{bmatrix}1&1\\0&0\end{bmatrix}+x_2\begin{bmatrix}1&0\\0&1\end{bmatrix}+x_3\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

Looking at the entry $(2,1)$ we see $x_3=0$.
So the set is LI.

LECTURE 20: DIMENSION AND RANK

PROBLEM 1: Show that any set of 4 vectors in $\mathbb{R}^3$ is linearly dependent.

Write the vectors as columns of a 3×4 matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} = A$$

By the definition of linear dependence, we need to find a NON ZERO $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

To find such $\vec{x}$, row reduce the matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 \end{bmatrix}$$

There are at most three pivots
so there is at least one free variable
therefore there is a non zero $\vec{x}$ in $\mathbb{R}^4$ such that $A\vec{x}=\vec{0}$,
that is, the columns of A are linearly dependent.

REMARK: any set with MORE than 3 vectors in $\mathbb{R}^3$ is LD. Similarly any subset of $\mathbb{R}^n$ with more than n vectors is LD.

A homogeneous linear system with more variables than equations has a nonzero solution, so the columns of a matrix that has more rows than columns ~~is~~ are LD.

THEOREM: let V be a vector space. If V has a basis with n vectors then all bases of V have n vectors.

~~The~~ **DEFINITION:** the ~~number of e~~ DIMENSION of a vector space is the number of elements in a base.

PROBLEM 2: prove that it is not possible to span $\mathbb{R}^3$ with only two vectors. In other words, any basis of $\mathbb{R}^3$ has at least 3 elements.

Assume $\mathbb{R}^3 = \text{span}\{\vec{v}, \vec{w}\}$. Then, there is a matrix

$$A = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} \quad \text{such that} \quad \begin{aligned} x_1 \vec{v} + x_2 \vec{w} &= \vec{e_1} \\ y_1 \vec{v} + y_2 \vec{w} &= \vec{e_2} \\ z_1 \vec{v} + z_2 \vec{w} &= \vec{e_3} \end{aligned}$$

We know that there are a, b, c not all zero such that

$$a \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + c \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

whence $a\vec{e_1} + b\vec{e_2} + c\vec{e_3} = \vec{0}$, contradiction.

REMARK: ~~any~~ If $m < n$, no set of m vectors can span $\mathbb{R}^n$.

03

EXAMPLES OF DIMENSION:

A) $\mathbb{R}^n$ has dimension n

B) $\mathbb{P}_n$ has dimension $n+1$

C) The space of $m\times n$ matrices has dimension $m \cdot n$.

DEFINITION: the RANK of a matrix is the dimension of its column space.

DEFINITION: the ROW SPACE of a matrix is the column space of its transpose.

EXERCISE: find bases for the row space and the column space of the matrix

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

THEOREM: in any matrix, the row space and the column space have the same dimension.

LECTURE 21: RANK (SECTION 4.6)

PROBLEM 1: Given two vectors $\vec{v}$ and $\vec{w}$ in the vector space V, show that $\text{Span}\{\vec{v}, \vec{w}\} = \text{Span}\{\vec{v}, \vec{v}+\vec{w}\}$.

THEOREM: Two row equivalent matrices have the same row space.

EXAMPLE: Find a basis for the row space of

$$\begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} = A$$

and write the rows of A as linear combinations of the elements of the basis.

Row reduce

$$A \sim \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 14 & -35 & 42 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: B$$

By the THM, A and B have the same row space, so

a basis is $\{(1,-4,9,-7), (0,-2,5,-6)\}$.

Besides $(-1,2,-4,1) = (0,-2,5,-6) - (1,-4,9,-7)$

and $(5,-6,10,7) = 5(1,-4,9,-7) - 7(0,-2,5,-6)$

EXAMPLE: find a basis for the column space of

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$$

THEOREM: in ~~any matrix~~ all matrices, the row space and the column space have the same dimension (= number of pivots).

EXAMPLE: draw the row space and the column space of

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

EXAMPLE: find the dimension of the null space of

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}.$$

THEOREM: dim Nul A + dim Col A = number of columns

LECTURE 22: EIGENVALUES AND EIGENVECTORS (SECTION 5.1)

Compute $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n$

DEFINE EIGENVALUES AND EIGENVECTORS

EXAMPLES

→ eigenvalues of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

→ $\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$ row reduce to prove that a, b and c are eigenvalues

→ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

→ any $\lambda > 0$ is an eigenvalue of the derivative operator

LECTURE 23: THE CHARACTERISTIC EQUATION (SECTION 5.2)

EIGENVECTORS AND EIGENVALUES

① The eigenvalues of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ are 1, 3 and -2, and the eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

② The eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are 1 and -1, and the eingenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

③ The eigenvalues of $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$ are $1, 3, -1$ and -3, and the eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$.

④ $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ does not have real eigenvalues.

HOW TO COMPUTE EIGENVALUES ?

λ is an eigenvalue of A ($n \times n$ matrix)

$\iff$ there is $v \in \mathbb{R}^n$, $v \neq 0$, such that $Av = \lambda v$

$\iff$ there is $v \in \mathbb{R}^n$, $v \neq 0$, such that $(A-\lambda)v = 0$

$\iff$ $A - \lambda I$ is NOT invertible

$\iff$ $\det(A - \lambda I) \neq 0$

DEFINITION: Let A be an $n \times n$ matrix. The CHARACTERISTIC POLYNOMIAL of A is the function

$$\lambda \longmapsto \det(A - \lambda I).$$

02

EXAMPLES:

(1) The characteristic polynomial of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is

$$\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)(-2-\lambda)$$

(2) The ch. poly. of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is

$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1.$$

(3) The ch. poly. of $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$ is

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 3 \\ 1 & 0 & -\lambda & 0 \\ 0 & 3 & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1-\lambda^2 & 0 \\ 0 & -\lambda & 0 & 3 \\ 1 & 0 & -\lambda & 0 \\ 0 & 3 & 0 & -\lambda \end{vmatrix} = (1-\lambda^2)\begin{vmatrix} 0 & -\lambda & 3 \\ 1 & 0 & 0 \\ 0 & 3 & -\lambda \end{vmatrix}$$

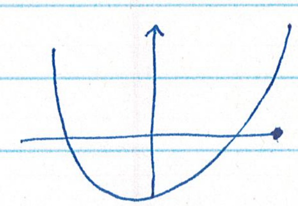
$$= -(1-\lambda^2)(\lambda^2-9)$$

$$= (\lambda^2-1)(\lambda^2-9).$$

~~SIMILAR MATRICES~~

Fact: a degree n polynomial ~~has~~ can be factored into n linear factors and the roots have MULTIPLICITIES.

$\lambda^2 - 4$ has roots -2 and 2



$\lambda^2 - 1$ has roots -1 and 1, both with "multiplicity" 1

λ^2 has only 0 as a root, with multiplicity 2.

$\lambda^3 - \lambda$ has roots $-1, 0$ and 1

$\lambda^3 - \lambda = (\lambda+1)\lambda(\lambda-1)$

$\lambda^3 + \lambda^2 = (\lambda+1)\lambda^2$ has roots -1 (mult. 1) and 0 (mult. 2)

λ^3 has root 0 with multiplicity 3

DEFINITION: A) Let $p(\lambda)$ be a polynomial. We say that r is a ROOT of $p(\lambda)$ of multiplicity k if $(\lambda-r)^k$ divides $p(\lambda)$ but $(\lambda-r)^{k+1}$ does not.

B) Let A be an ~~m~~ $n \times n$ matrix. We say that r is an eigenvalue of A with multiplicity k if $(\lambda-r)^k$ divides $\det(A-\lambda I)$ but $(\lambda-r)^{k+1}$ does not.

EXAMPLE: $\begin{pmatrix} 7 & -4 & 0 & 6 \\ 0 & 4 & 1 & 6 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 8 \end{pmatrix}$ has eigenvalues 4 (mult. 1), 7 (mult. 2) and 8 (mult. 1).

~~PROBLEM:~~

FACT: if ~~the~~ an eigenvalue has multiplicity k then its eigenspace has dimension at least k.

LECTURE 23: DIAGONALIZATION (SECTION 5.3)

A vector equation is a shortcut for several numeric equations. In the same way, a matrix equation is a shortcut for several ~~Sco~~ vector equations.

Recall $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1.

The equations

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

can be written as a single matrix equation

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

If A is an $n \times n$ matrix and

$$A v_1 = \lambda_1 v_1$$
$$A v_2 = \lambda_2 v_2$$
$$\vdots$$
$$A v_n = \lambda_n v_n$$

then

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

DEFINITION: A $n \times n$ matrix is DIAGONALIZABLE if $\mathbb{R}^n$ has a basis of eigenvectors of A.

Then $AP = PD$, where

.) the columns of P are eigenvectors of A

..) ~~entry~~ D is a diagonal matrix

02

If a matrix is diagonalizable then it is possible to ~~do~~ compute polynomials of this matrix (such as $A^3+2A+3I$) easily.

<u>EXAMPLES</u>: diagonalize the matrices and write them in the form $A = P^{-1}DP$, with P diagonal.

*) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

**) $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

1) Find the eigenvalues.

Characteristic equation: $0 = \begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = \lambda^2 - 1$

$\Rightarrow$ eigenvalues $+1$ and -1

2) Find the eigenvectors

2a) Eigenv. associated to 1: solve for $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\Longleftrightarrow \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Pick $\underbrace{\text{one solution}}_{\text{non-zero}}$, for example $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2b) Eigenv. associated to -1: solve for $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\Longleftrightarrow \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Pick $\underbrace{\text{one solution}}_{\text{non-zero}}$, for example $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3) Combine the equations $Av_i = \lambda_i v_i$ into a single matrix equation, and CHECK.

$$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

4) Verify if the eigenvectors are LI (i.e. if P is invertible). If they are not then the matrix is NOT diagonalizable.

**) $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} = A$

1) Find the eigenvalues. Those are 4, 4 and 5.

2) Are there two LI eigenvectors for 4 ?

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is an eigenvector associated to 4

$\Longleftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ Ⓐ

Two LI eigenvectors $\Longleftrightarrow$ two free variables in Ⓐ.

There are two pivots, so only one free variable.

So A is NOT DIAGONALIZABLE.

THEOREM: Let A be an $n \times n$ matrix.

*) If A has n distinct eigenvalues then A is diagonalizable.

**) If A is symmetric then A is diagonalizable.

To see why (*) is true, assume the eigenvalues are $\lambda_1, \dots, \lambda_n$ and pick eigenvectors $v_1, \dots, v_n$. Are $v_1, \dots, v_n$ LI ?

* If $x_1 v_1 + \dots + x_n v_n = 0$ then

$x_1 \lambda_1 v_1 + \dots + x_n \lambda_n v_n = 0$

$\Longrightarrow x_2(\lambda_2 - \lambda_1) v_2 + \dots + x_n(\lambda_n - \lambda_1) v_n = 0$

LECTURE 24: EIGENVECTORS AND LINEAR TRANSFORMATIONS

DEFINITION: If $B = (\overset{v_1}{\cancel{b_1}}, \dots, \overset{v_n}{\cancel{b_n}})$ is a basis of the vector space V and $v \in V$ then the COORDINATE VECTOR of v relative to B is

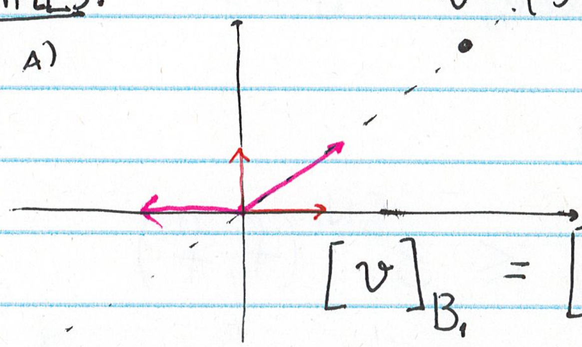
$$[v]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{where} \quad v = a_1 v_1 + \dots + a_n v_n .$$

EXAMPLES:

A) $v = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$B_1 = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

$B_2 = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$



$$[v]_{B_1} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$[v]_{B_2} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

B) Bases of $\mathbb{P}_3$

$B_1 = (1, t, t^2, t^3)$

$B_2 = (1, 1+t, t+t^2, t^3)$

$$p(t) = 3t^2 + t^3 = t^3 + 3(t^2+t) - 3(t+1) + 3$$

$$[p]_{B_1} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} \qquad [p]_{B_2} = \begin{bmatrix} 3 \\ -3 \\ 3 \\ 1 \end{bmatrix}$$

02

Recall a linear transformation $A: V \to W$ between vector spaces V and W is a function that satisfies

$$A(v_1 + v_2) = Av_1 + Av_2$$

$$A(cv) = cAv$$

The MATRIX OF A RELATIVE TO THE BASES B_V and B_W is ~~the~~.

$$\begin{bmatrix} | & | & & | \\ [Av_1]_W & [Av_2]_W & \cdots & [Av_n]_W \\ | & | & & | \end{bmatrix}$$

EXAMPLES: A) ~~Basis of~~ reflection through $x=y$

B) differentiation

03

PROBLEM: $T: \mathbb{P}_2 \to \mathbb{P}_2$ is defined by

$$T(a_0 + a_1 t + a_2 t^2) = 4a_0 + (3a_0 - 9a_1)t + (2a_1 + 6a_2)t^2.$$

A) Is T a linear transformation?

B) If so, find the matrix representation of T in the basis $\mathcal{B} = \{1, t, t^2\}$

A) Need to check

$$T(ca_0 + ca_1 t + ca_2 t^2) \stackrel{?}{=} c\,T(a_0 + a_1 t + a_2 t^2)$$

$$T\left((a_0+b_0) + (a_1+b_1)t + (a_2+b_2)t^2\right) \stackrel{?}{=} T(a_0 + a_1 t + a_2 t^2) + \\ + T(b_0 + b_1 t + b_2 t^2)$$

B) Both the domain and the codomain have dimension 3, so the matrix of T is 3×3.

Column 1 has the coefficients of $T(1)$ in the basis $(1, t, t^2)$, column 2 has the coefficients of $T(t)$, column 3 has the coefficients of $T(t^2)$.

$$T(1) = 4 + 3t + 0 \cdot t^2$$

$$T(t) = 0 + (-9)t + 2 \cdot t^2$$

$$T(t^2) = 0 + 0 \cdot t + 6 \cdot t^2$$

The matrix of T in the basis $\{1, t, t^2\}$ is

$$\begin{bmatrix} 4 & 0 & 0 \\ 3 & -9 & 0 \\ 0 & 2 & 6 \end{bmatrix}$$

PROBLEM: the linear operator $A: \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}.$$

*) Compute $A\begin{bmatrix}5\\6\end{bmatrix}$.

**) What is the matrix of A relative to the basis $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$?

***) What is the matrix of A relative to the basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$?

*) If we find x, y such that $\begin{bmatrix}5\\6\end{bmatrix} = x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix}$

then $A\begin{bmatrix}5\\6\end{bmatrix} = xA\begin{bmatrix}1\\1\end{bmatrix} + yA\begin{bmatrix}-1\\1\end{bmatrix} = x\begin{bmatrix}2\\3\end{bmatrix} + y\begin{bmatrix}4\\5\end{bmatrix}$

$$\left.\begin{aligned} 5 &= x - y \\ 6 &= x + y \end{aligned}\right\} \qquad x = \frac{11}{2} \qquad y = \frac{1}{2}$$

$$\Rightarrow A\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}13\\19\end{bmatrix}.$$

**) $\begin{bmatrix}2 & 4\\3 & 5\end{bmatrix}$

***) Need to find $A\begin{bmatrix}1\\0\end{bmatrix}$ and $A\begin{bmatrix}0\\1\end{bmatrix}$.

Let x_1, y_1, x_2, y_2 be such that

$$\begin{bmatrix}1\\0\end{bmatrix} = x_1\begin{bmatrix}1\\1\end{bmatrix} + y_1\begin{bmatrix}-1\\1\end{bmatrix} \qquad \begin{bmatrix}0\\1\end{bmatrix} = x_2\begin{bmatrix}1\\1\end{bmatrix} + y_2\begin{bmatrix}-1\\1\end{bmatrix}$$

05

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

The matrix of A in the basis $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is

$$\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

LECTURE 26: COMPLEX EIGENVALUES (SECTION 5.5)

SUMMARY:

A) Complex eigenvalues come in conjugate pairs.
(if the matrix has only real entries)

B) If a matrix has a complex eigenvalue then it has an invariant subspace of dimension 2.

REAL EIGENVALUES $\rightsquigarrow$ dilation

COMPLEX EIGENVALUES $\rightsquigarrow$ dilation + rotation

LECTURE 26: COMPLEX EIGENVALUES (SECTION 5.5)

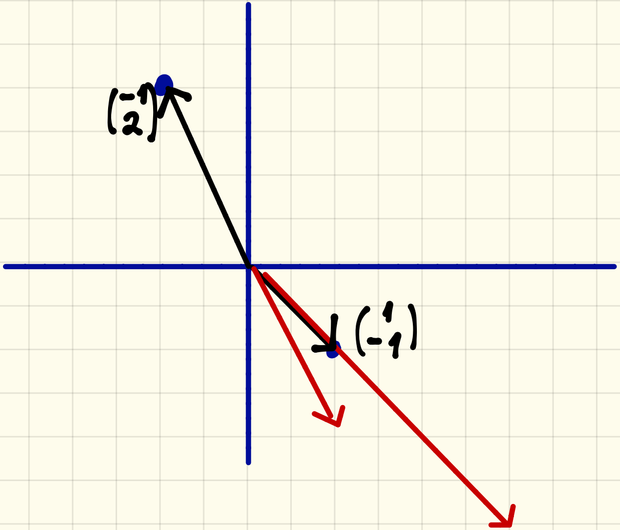
GOAL: given a $n \times n$ matrix A, find a basis of $\mathbb{R}^n$ formed by eigenvectors of A.

In other words, diagonalize A, i.e. factor

$$A = P \underbrace{D}_{\text{diagonal}} P^{-1}$$

EXAMPLES

$$\begin{pmatrix} 7 & 4 \\ -8 & -5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$



Some matrices are NOT diagonalizable

The matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ has eigenvalues $a+bi$ and $a-bi$

eigenvectors $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ +i \end{pmatrix}$.

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} a-bi \\ b+ai \end{pmatrix} = \begin{pmatrix} (a-bi)1 \\ (a-bi)i \end{pmatrix}$$

Some matrices are NOT diagonalizable

The matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = A$$

has eigenvalues 1, 1, 1 and is NOT diagonalizable.

There are no 3 LI eigenvectors

$Av = v$ has only one free variable

GOAL: given a $n \times n$ matrix A, factor

$$A = PNP^{-1}$$

for a block diagonal matrix N.

EXAMPLE OF NICE MATRIX

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & 2 & -3 \end{bmatrix}$$

Eigenvalues: $3, 5, 1+2i, 1-2i, -3+2i, -3-2i$

Eigenvectors: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ i \end{pmatrix}$

a eigenv $\rightsquigarrow A v = \lambda v$

$a \pm bi \rightsquigarrow A(\mathrm{span}\{v, w\}) = \mathrm{span}\{v, w\}$

THEOREM A $n \times n$ square matrix with real entries.
If $a+bi$ is an eigenvalue of A than $a-bi$ is also an eigenvalue.

Because if $a+bi$ is an eigenvalue of A
then there is $v = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ such that $Av = (a+bi)v$

Take the conjugate $\overline{a+bi} := a-bi$. Notice $\overline{zw} = \bar{z}\,\bar{w}$

$$Av = (a+bi)v \Rightarrow \overline{Av} = \overline{(a+bi)v} = (a-bi)\bar{v}$$

$\overline{Av} = A\bar{v}$ because the entries of A are real

$$\bar{v} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{bmatrix}$$

$\Rightarrow \bar{v}$ is an eigenvector with eigenvalue $a-bi$.

If z is a solution of

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad \leftarrow a_n, a_{n-1}, \cdots, a_0 \text{ in } \mathbb{R}$$

then $\bar{z}$ also a solution.

Because

if $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$

then

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = 0$$

$$\| \leftarrow \overline{z+w} = \bar{z} + \bar{w}$$

$$\overline{a_n z^n} + \cdots + \overline{a_1 z} + \overline{a_0} = a_n \bar{z}^n + \cdots + a_1 \bar{z} + a_0$$

THEOREM: if $a+bi$ and $a-bi$ are eigenvalues of A and $b \neq 0$ then A has an invariant subspace of dimension 2.

That is, there are v, w in $\mathbb{R}^n$ LI such that

$$A(\operatorname{span}\{v,w\}) = \operatorname{span}\{v,w\}$$

$\widetilde{a+bi}$ is an eigenvalue of A

$\Rightarrow$ there are v, w in $\mathbb{R}^n$ such that

$$A(v+iw) = (a+bi)(v+iw)$$

$$Av + iAw = (av-bw) + i(bv+aw)$$

$$\Rightarrow Av = av - bw$$

$$Aw = bv + aw$$

$\Rightarrow A(xv+yw) = \tilde{x}v + \tilde{y}w$ for some $\tilde{x}, \tilde{y}$

$a+bi$ is an eigenvalue of A

$\Rightarrow$ there are v, w in $\mathbb{R}^n$ such that

$$Av = av - bw$$

$$Aw = bv + aw$$

Why are v and w LI?

If $v = \lambda w$ for some λ in $\mathbb{R}$ then

$$A(v+iw) = (a+ib)(\lambda+i)w \quad \text{(I)}$$

and

$$Aw = (b\lambda+a)w \quad \text{(II)}$$

From (II),

$$A(v+iw) = (\lambda+i)(b\lambda+a)w \quad \text{(III)}$$

(I) = (III): $(b\lambda+a)w = (a+ib)w$

$\Rightarrow$ either $w=0$ or $b=0$, contradiction.

A) Complex eigenvalues come in conjugate pairs.

B) If a matrix has a complex eigenvalue then it has an invariant subspace of dimension 2.

REAL EIGENVALUES $\rightsquigarrow$ dilation

COMPLEX EIGENVALUES $\rightsquigarrow$ dilation + rotation

What is a differential equation?

1) Find all functions

$$x : [0, +\infty) \to \mathbb{R}$$

such that $\boxed{x'(t) = 0}$ for all t.

In other words: if you move always at speed zero, how far from your starting point are you at time t?

The constant functions are solutions and only the constant functions.

What is a differential equation?

2) Find all functions

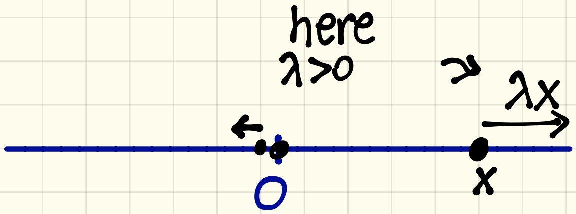
$$y: \mathbb{R} \to \mathbb{R}$$

such that $\boxed{y'(t) = \lambda y(t)}$ for all t.

$$y(t) = y(0)e^{\lambda t}$$

$y(0)$ is a free parameter

A point moves on a line. When its position is x, its speed is λx.

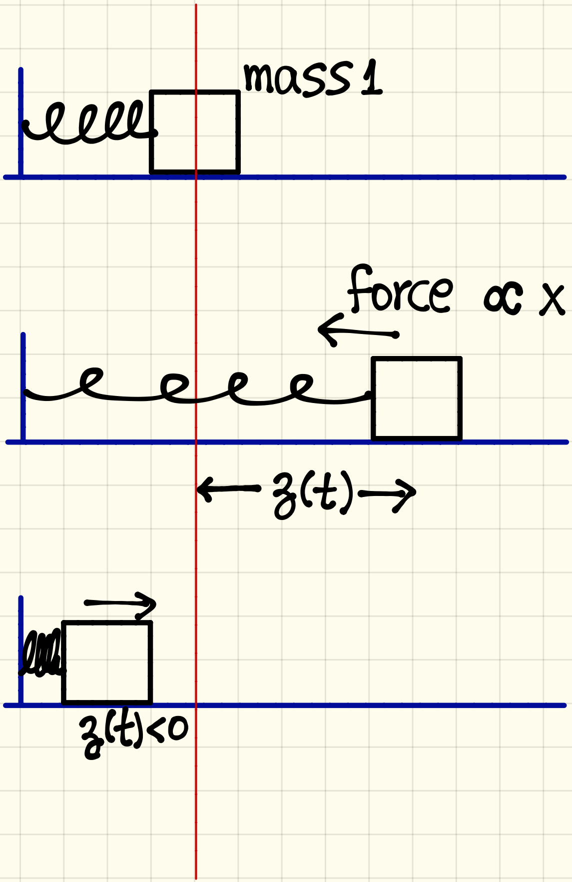


What is a differential equation?

3) Find all functions

$$z : \mathbb{R} \to \mathbb{R}$$

such that $\boxed{z''(t) = -z(t)}$ for all t.



A solution is

$$z_1(t) = \lambda \cos t \qquad z_1''(t) = -z_1(t)$$

Another solution is

$$z_2(t) = \sin t \qquad z_2''(t) = -z_2(t)$$

Also $z(t) = 0$.

More solutions: $A \sin t + B \cos t$ for any A, B.

What is a LINEAR differential equation?

A linear ODE is an equation that satisfies the

PRINCIPLE OF SUPERPOSITION:

linear combinations of solutions are also solutions

the set of solutions is a vector space

EXAMPLES

•) The equation $ay''' + by'' + cy = 0$ is linear.

••) The equation $ay''' + by'' + cy = 1$ is NOT linear.

Say y_1 and y_2 are solutions. Then

$$a(y_1 + y_2)''' + b(y_1 + y_2)'' + c(y_1 + y_2) = 2$$

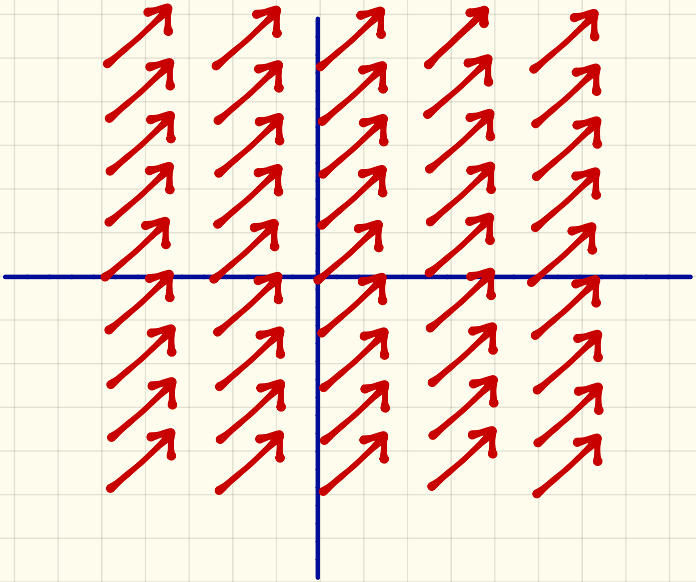
∴) The equation $y'(x) = x \cdot y(x) + y(x)^2$ is NOT linear.

If y solves the equation and $y \neq 0$ then $2y$ does NOT solve the equation.

2D systems of ODEs

Given $F: \mathbb{R}^2 \to \mathbb{R}^2$ vector field

$F(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



2D systems of ODEs

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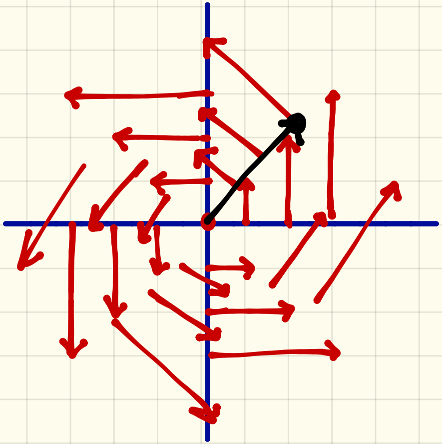


$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = 90^\circ \circlearrowleft \text{ of } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$i(x_1 + i x_2) = -x_2 + i x_1$$

2D systems of ODEs

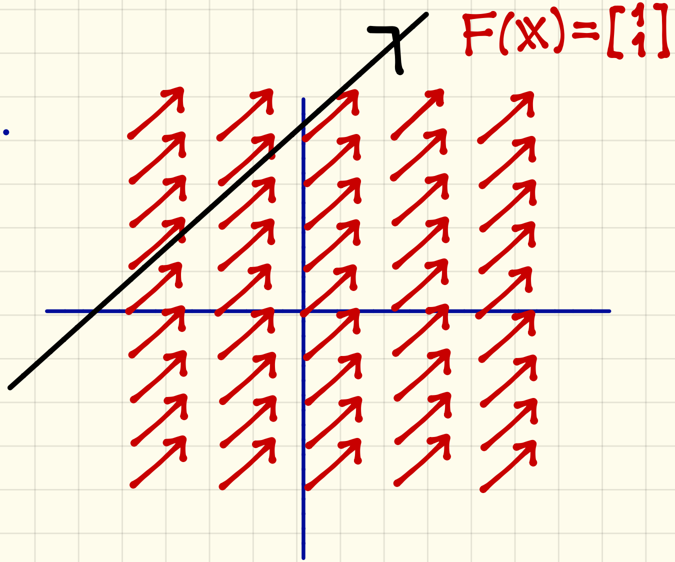
Given $F:\mathbb{R}^2 \to \mathbb{R}^2$ vector field

Find all functions $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ such that

$$\boxed{\mathbf{x}'(t) = F(\mathbf{x}(t))} \text{ for all } t.$$

A point is moving on the plane.

At position $\mathbf{x}$ its speed vector is $F(\mathbf{x})$.



Solutions to $\mathbf{x}'(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} a \\ b \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2D systems of ODEs

Given $F: \mathbb{R}^2 \to \mathbb{R}^2$ vector field

Find all functions $\mathbb{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ such that

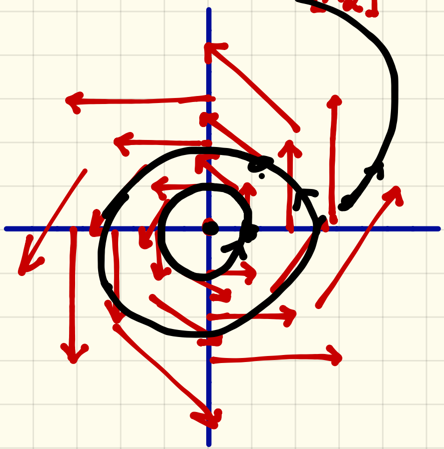
$$\boxed{\mathbb{x}'(t) = F(\mathbb{x}(t))}$$ for all t.

$$\mathbb{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbb{x}(t)$$

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

A point is moving on the plane.

At position $\mathbb{x}$ its speed vector is $F(\mathbb{x})$.



some Solutions of $\mathbb{x}'(t) = \begin{bmatrix} -x_2(t) \\ x_1(t) \end{bmatrix}$

$$t \mapsto R\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

2D first-order LINEAR systems of ODEs

Given A 2×2 matrix

Find all functions $\mathbb{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ such that

$$\boxed{\mathbb{x}'(t) = A(\mathbb{x}(t))}$$ for all t.

EXAMPLE $\quad \mathbb{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbb{x}(t)$

Solving $\mathbf{x}' = A\mathbf{x}$ when A is diagonal

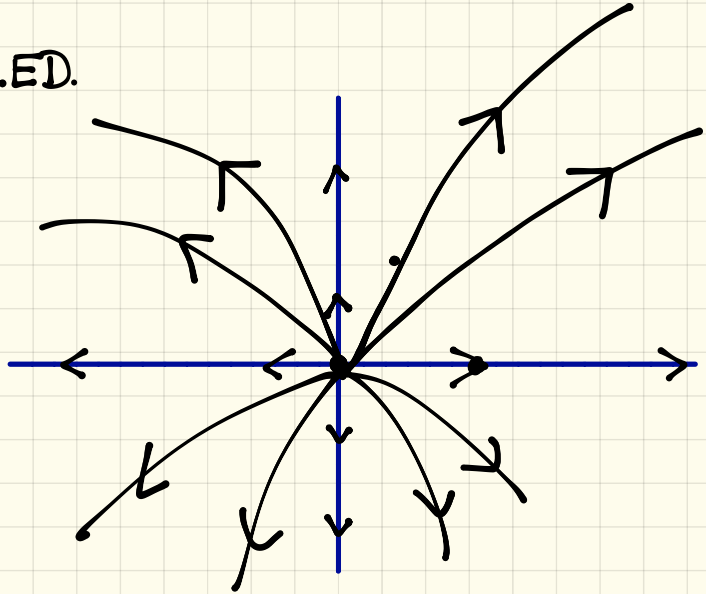
$x_1'(t) = 3x_1(t)$
$x_2'(t) = 5x_2(t)$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This system is DECOUPLED.

$x_1(t) = x_1(0)e^{3t}$

$x_2(t) = x_2(0)e^{5t}$



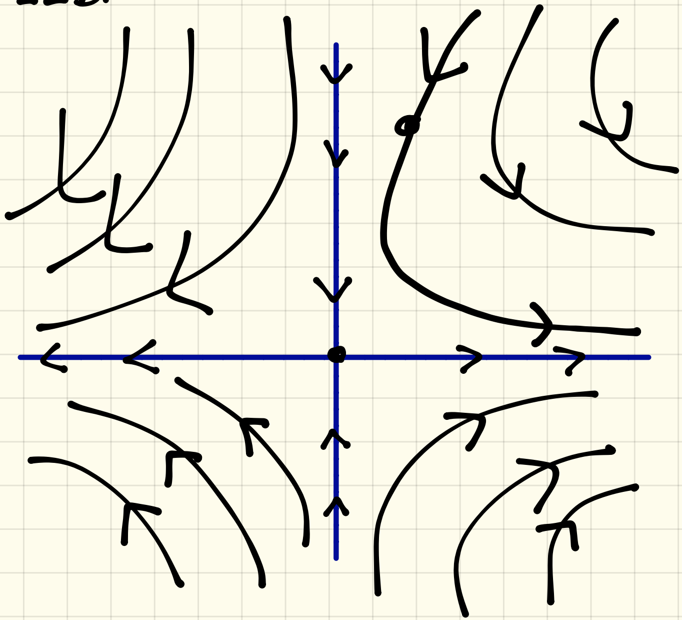
Solving $\mathbf{x}' = A\mathbf{x}$ when A is diagonal

$$x_1'(t) = 3x_1(t)$$
$$x_2'(t) = -5x_2(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This system is DECOUPLED.

$$x_1(t) = x_1(0)e^{3t}$$
$$x_2(t) = x_2(0)e^{-5t}$$



LECTURE 29: SOME EXAMPLES OF LINEAR ODEs (SECTION 5.7)

GOAL: given a 2x2 matrix A

solve the equation $\underbrace{\mathbf{x}}_{}'(t) = A\mathbf{x}(t)$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We know how to solve $\mathbf{y}'(t) = D\mathbf{y}(t)$ $\quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$\mathbf{y}(t) = \begin{bmatrix} e^{\lambda_1 t} y_1(0) \\ e^{\lambda_2 t} y_2(0) \end{bmatrix} \qquad \mathbf{y}(t) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$$

Solving $\mathbf{x}' = A\mathbf{x}$ when A is diagonalizable

$$x_1'(t) = 2x_1(t) + 3x_2(t)$$
$$x_2'(t) = -x_1(t) - 2x_2(t)$$

$$\mathbf{x}'(t) = \underbrace{\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}}_{A} \mathbf{x}(t)$$

Suppose $A = PDP^{-1}$ with D diagonal matrix.

We know how to solve $\mathbf{y}'(t) = D\mathbf{y}(t)$

Want to solve $\mathbf{x}'(t) = PDP^{-1}\mathbf{x}(t)$

$$\Rightarrow \underbrace{P^{-1}\mathbf{x}'(t)}_{=(P^{-1}\mathbf{x}(t))'} = DP^{-1}\mathbf{x}(t)$$

Let $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

To solve $\mathbf{x}'(t) = A\mathbf{x}(t)$

1. Diagonalize $A = PDP^{-1}$
2. Solve $\mathbf{y}'(t) = D\mathbf{y}(t)$.
3. Solve $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

$$\mathbf{x}'(t) = \underbrace{\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}}_{A} \mathbf{x}(t)$$

To solve $\mathbf{x}'(t) = A\mathbf{x}(t)$

① Diagonalize $A = PDP^{-1}$

② Solve $\mathbf{y}'(t) = D\mathbf{y}(t)$.

③ Solve $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

① Eigenvalues? 1 and -1

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = \cdots = (\lambda+1)(\lambda-1)$$

Eigenvectors?

By definition, the eigenvectors $\begin{bmatrix} x \\ y \end{bmatrix}$ associated to 1

$$(A - 1\cdot I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff A\begin{bmatrix} x \\ y \end{bmatrix} = 1\cdot\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Pick an eigenvector, e.g. $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Similarly $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector associated to -1

$$\Longrightarrow A = \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}^{-1}$$

$$\mathbf{x}'(t) = \underbrace{\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}}_{A} \mathbf{x}(t)$$

To solve $\mathbf{x}'(t) = A\mathbf{x}(t)$

① Diagonalize $A = PDP^{-1}$

② Solve $\mathbf{y}'(t) = D\mathbf{y}(t)$.

③ Solve $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

$$A = \underbrace{\begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}^{-1}$$

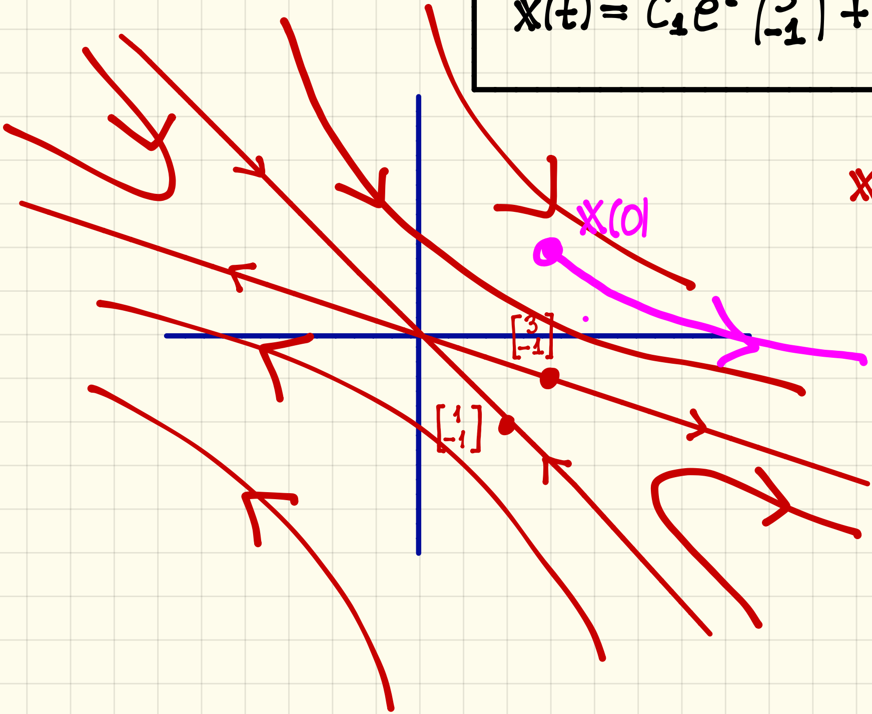
② $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{t} y_1(0) \\ e^{-t} y_2(0) \end{pmatrix}$ free

③ Let c_1, c_2 be numbers. $\mathbf{y}(t) = \begin{pmatrix} c_1 e^{t} \\ c_2 e^{-t} \end{pmatrix}$

Solve $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

$$\Rightarrow P\mathbf{y}(t) = \mathbf{x}(t) \Rightarrow \boxed{\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$\mathbf{x}(t) = c_1 e^{t}\begin{pmatrix}3\\-1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$$



$\mathbf{x}' = A\mathbf{x}$

Solve the initial value problem

$$x_1'(t) = 2x_1(t) + 3x_2(t) \qquad x_1(0) = 3$$
$$x_2'(t) = -x_1(t) - 2x_2(t) \qquad x_2(0) = 2$$

Ⓐ Plug $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ into $\boxed{\mathbf{x}(t) = C_1 e^{t}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

and solve for C_1 and C_2.

Ⓑ Go back to the solution

$$\mathbf{x}(t) = P\,\mathbf{y}(t)$$

Solving $\mathbf{x}' = A\mathbf{x}$ when A has complex eigenvectors

$$x_1'(t) = a x_1(t) - b x_2(t) \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$x_2'(t) = b x_1(t) + a x_2(t)$$

FACT: the space of solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ has dimension 2.

CONSEQUENCE: If $\mathbf{x}'(t) = A\mathbf{x}(t)$, $\mathbf{y}'(t) = A\mathbf{y}(t)$ and $\{\mathbf{x}(t), \mathbf{y}(t)\}$ is LI then ALL solutions are linear combinations of $\mathbf{x}(t)$ and $\mathbf{y}(t)$:

$$C_1 \mathbf{x}(t) + C_2 \mathbf{y}(t).$$

If we find two LI solutions, the problem is solved.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If we find two LI solutions, the problem is solved.

$$\mathbf{x}' = A\mathbf{x}$$

How do we find two LI solutions?

How do we find ANY solution?

We know that A has eigenvalues $a+ib$ and $a-ib$,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = (a+bi) \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = (a-bi) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\mathbf{x}' = A\mathbf{x}$ If we find two LI solutions, the problem is solved.

How do we find two LI solutions?

Assume we have $A\mathbf{v} = z\mathbf{v}$ and z is not real (i.e. $z = a + ib$, $b \neq 0$).

Then a complex solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = e^{tz}\mathbf{v}.$$

So the real and imaginary parts of $e^{tz}\mathbf{v}$ are solutions of $\mathbf{x}' = A\mathbf{x}$. Are they LI?

Yes!

① eigenvectors associated to $\neq$ eigenvalues are LI

② $\operatorname{Re} w = \frac{1}{2}(w + \bar{w})$ $\quad \operatorname{Im} w = \frac{1}{2i}(w - \bar{w}) = \frac{-i}{2}(w - \bar{w})$

$$\left(\underbrace{\operatorname{Re} e^{tz}\mathbf{v} \quad \operatorname{Im} e^{tz}\mathbf{v}}_{2\times 2\text{ matrix}}\right) = \left(\underbrace{e^{tz}\mathbf{v} \quad e^{t\bar{z}}\bar{\mathbf{v}}}_{\text{invertible}}\right)\frac{1}{2}\underbrace{\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}}_{\text{invertible}}$$

If $A\mathbb{v} = z\mathbb{v}$ and z is not real,
take $\mathrm{Re}(e^{tz}\mathbb{v})$ and $\mathrm{Im}(e^{tz}\mathbb{v})$

$$\underset{A}{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} \underset{\mathbb{v}}{\begin{pmatrix} 1 \\ -i \end{pmatrix}} = \underset{z}{(a+bi)} \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = (a-bi) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$e^{tz}\mathbb{v} = e^{at}(\cos bt + i \sin bt) \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= e^{at} \begin{pmatrix} \cos bt \\ \sin bt \end{pmatrix} + i e^{at} \begin{pmatrix} \sin bt \\ -\cos bt \end{pmatrix}$$

If $A\mathbf{v} = z\mathbf{v}$ and z is not real, take $\mathrm{Re}(e^{tz}\mathbf{v})$ and $\mathrm{Im}(e^{tz}\mathbf{v})$

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$e^{tz}\mathbf{v} = e^{at}\begin{pmatrix} \cos bt \\ \sin bt \end{pmatrix} + i e^{at}\begin{pmatrix} \sin bt \\ -\cos bt \end{pmatrix}$$

ANSWER: all the solutions of the system $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ have the form

$$\mathbf{x}(t) = C_1 e^{at}\begin{pmatrix} \cos bt \\ \sin bt \end{pmatrix} + C_2 e^{at}\begin{pmatrix} \sin bt \\ -\cos bt \end{pmatrix}$$

for some choice of numbers C_1 and C_2.

What is a differential equation?

1) Find all functions

$$x : [0, +\infty) \to \mathbb{R}$$

such that $\boxed{x'(t) = 0}$ for all t.

In other words: if you move always at speed zero, how far from your starting point are you at time t?

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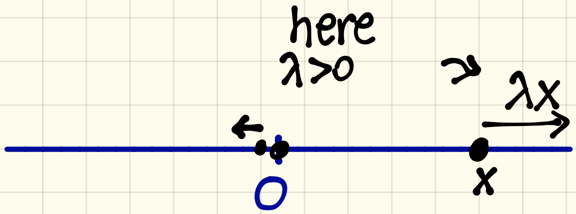

What is a differential equation?

3) Find all functions

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such that $\boxed{z''(t) = -z(t)}$ for all t.

A solution is

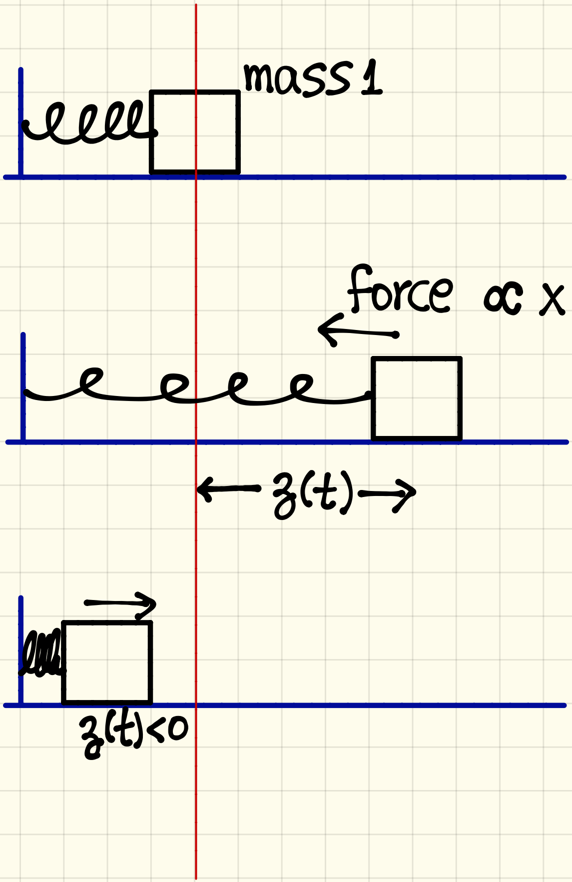
$$z_1(t) = \lambda \cos t \qquad z_1''(t) = -z_1(t)$$

Another solution is

$$z_2(t) = \sin t \qquad z_2''(t) = -z_2(t)$$

Also $z(t) = 0$.

More solutions: $A \sin t + B \cos t$ for any A, B.



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the set of solutions is a vector space

EXAMPLES

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••) The equation $ay''' + by'' + cy = 1$ is NOT linear.

Say y_1 and y_2 are solutions. Then

$$a(y_1 + y_2)''' + b(y_1 + y_2)'' + c(y_1 + y_2) = 2$$

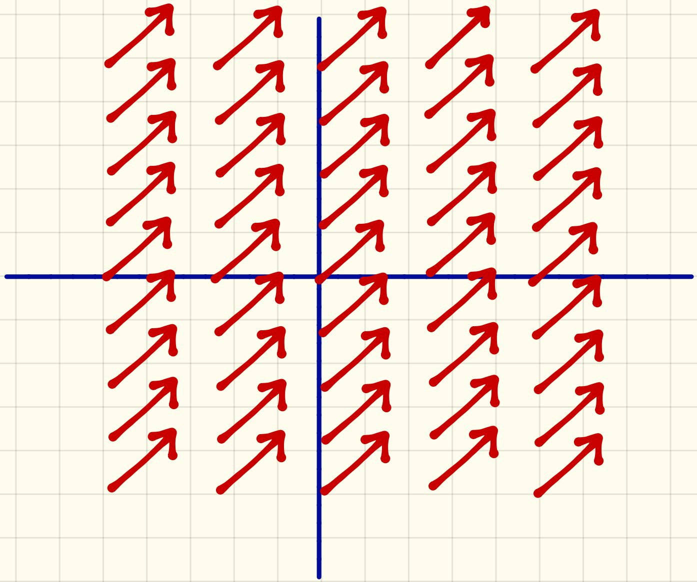
∴) The equation $y'(x) = x \cdot y(x) + y(x)^2$ is NOT linear.

If y solves the equation and $y \neq 0$ then $2y$ does NOT solve the equation.

2D systems of ODEs

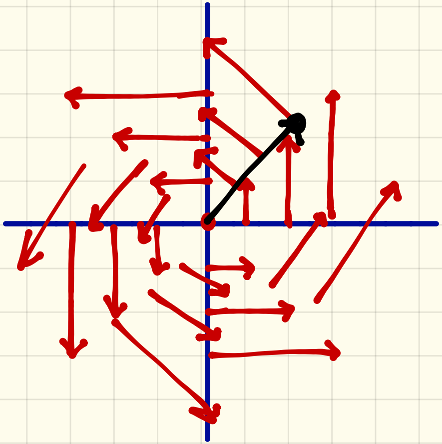
Given $F: \mathbb{R}^2 \to \mathbb{R}^2$ vector field

$$F(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



2D systems of ODEs

Given $F: \mathbb{R}^2 \to \mathbb{R}^2$ vector field



$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = 90^\circ \circlearrowleft \text{ of } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$i(x_1 + i x_2) = -x_2 + i x_1$$

2D systems of ODEs

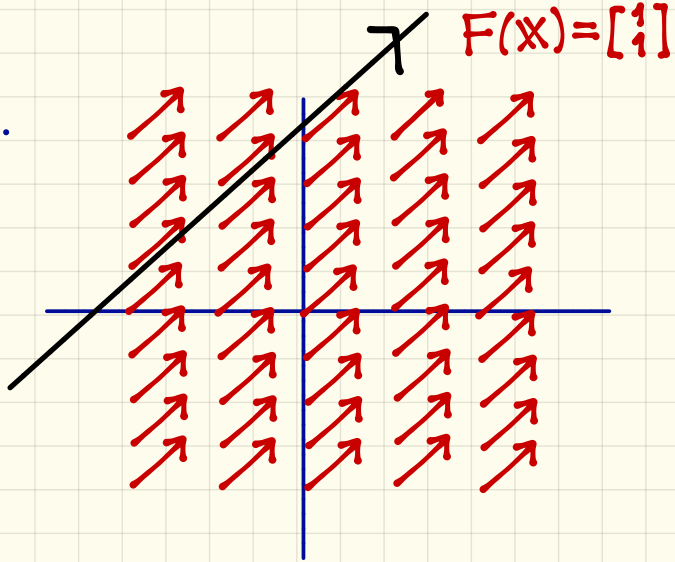
Given $F: \mathbb{R}^2 \to \mathbb{R}^2$ vector field

Find all functions $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ such that

$$\boxed{\mathbf{x}'(t) = F(\mathbf{x}(t))} \text{ for all } t.$$

A point is moving on the plane.

At position $\mathbf{x}$ its speed vector is $F(\mathbf{x})$.



Solutions to $\mathbf{x}'(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} a \\ b \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2D systems of ODEs

Given $F:\mathbb{R}^2 \to \mathbb{R}^2$ vector field

Find all functions $\mathbb{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ such that

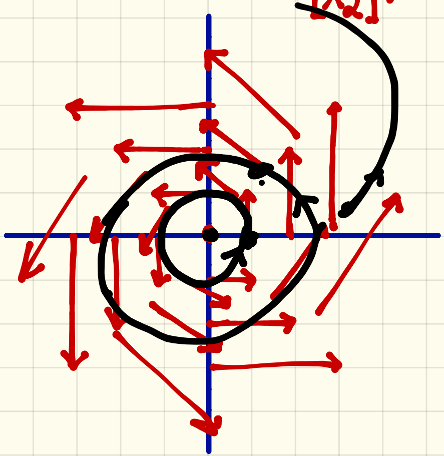
$$\boxed{\mathbb{x}'(t) = F(\mathbb{x}(t))} \quad \text{for all } t.$$

$$\mathbb{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbb{x}(t)$$

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

A point is moving on the plane.

At position $\mathbb{x}$ its speed vector is $F(\mathbb{x})$.



some Solutions of $\mathbb{x}'(t) = \begin{bmatrix} -x_2(t) \\ x_1(t) \end{bmatrix}$

$$t \mapsto R\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

2D first-order LINEAR systems of ODEs

Given A 2×2 matrix

Find all functions $\mathbb{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ such that

$$\boxed{\mathbb{x}'(t) = A(\mathbb{x}(t))} \text{ for all } t.$$

EXAMPLE $\quad \mathbb{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbb{x}(t)$

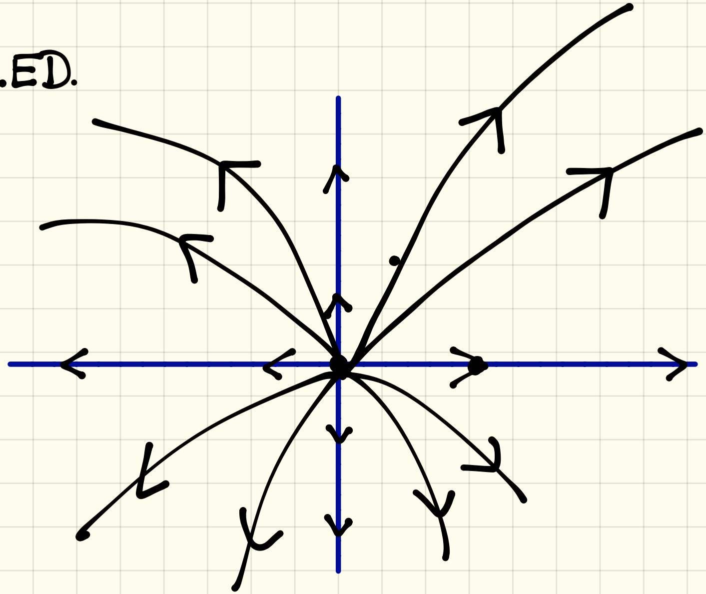
Solving $\mathbf{x}' = A\mathbf{x}$ when A is diagonal

$$x_1'(t) = 3x_1(t)$$
$$x_2'(t) = 5x_2(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This system is DECOUPLED.

$$x_1(t) = x_1(0)e^{3t}$$
$$x_2(t) = x_2(0)e^{5t}$$



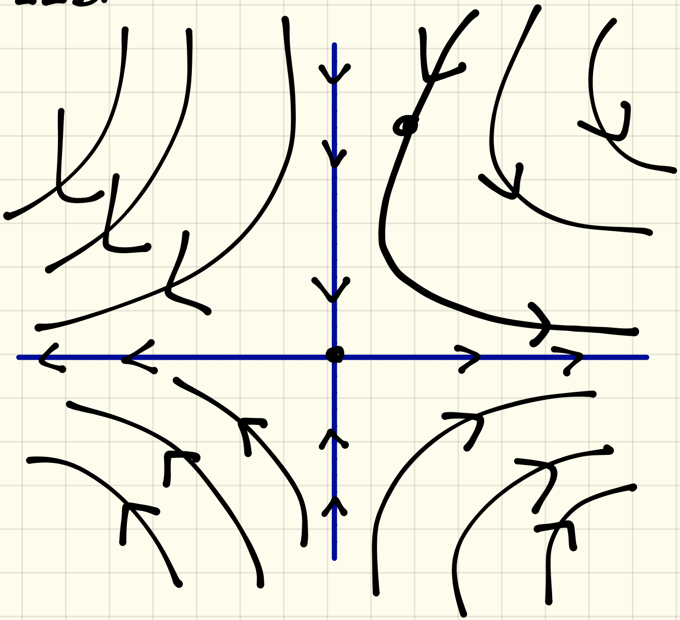
Solving $\mathbf{x}' = A\mathbf{x}$ when A is diagonal

$$x_1'(t) = 3x_1(t)$$
$$x_2'(t) = -5x_2(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This system is DECOUPLED.

$$x_1(t) = x_1(0)e^{3t}$$
$$x_2(t) = x_2(0)e^{-5t}$$



LECTURE 29: SOME EXAMPLES OF LINEAR ODEs (SECTION 5.7)

GOAL: given a 2×2 matrix A

solve the equation $\underbrace{\mathbf{x}'(t)}= A\mathbf{x}(t)$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We know how to solve $\mathbf{y}'(t) = D\mathbf{y}(t)$ $\quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$\mathbf{y}(t) = \begin{bmatrix} e^{\lambda_1 t} y_1(0) \\ e^{\lambda_2 t} y_2(0) \end{bmatrix} \qquad \mathbf{y}(t) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$$

Solving $\mathbf{x}' = A\mathbf{x}$ when A is diagonalizable

$$x_1'(t) = 2x_1(t) + 3x_2(t)$$
$$x_2'(t) = -x_1(t) - 2x_2(t)$$

$$\mathbf{x}'(t) = \underbrace{\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}}_{A} \mathbf{x}(t)$$

Suppose $A = PDP^{-1}$ with D diagonal matrix.

We know how to solve $\mathbf{y}'(t) = D\mathbf{y}(t)$

Want to solve $\mathbf{x}'(t) = PDP^{-1}\mathbf{x}(t)$

$$\Rightarrow \underbrace{P^{-1}\mathbf{x}'(t)}_{=(P^{-1}\mathbf{x}(t))'} = DP^{-1}\mathbf{x}(t)$$

Let $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

To solve $\mathbf{x}'(t) = A\mathbf{x}(t)$

① Diagonalize $A = PDP^{-1}$

② Solve $\mathbf{y}'(t) = D\mathbf{y}(t)$.

③ Solve $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

$$\mathbf{x}'(t) = \underbrace{\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}}_{A} \mathbf{x}(t)$$

To solve $\mathbf{x}'(t) = A\mathbf{x}(t)$

① Diagonalize $A = PDP^{-1}$

② Solve $\mathbf{y}'(t) = D\mathbf{y}(t)$.

③ Solve $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

① Eigenvalues? 1 and -1

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = \cdots = (\lambda+1)(\lambda-1)$$

Eigenvectors?

By definition, the eigenvectors $\begin{bmatrix} x \\ y \end{bmatrix}$ associated to 1

$$(A - 1\cdot I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff A\begin{bmatrix} x \\ y \end{bmatrix} = 1\cdot\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Pick an eigenvector, e.g. $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Similarly $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector associated to -1

$$\Longrightarrow A = \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}^{-1}$$

$$\mathbf{x}'(t) = \underbrace{\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}}_{A} \mathbf{x}(t)$$

To solve $\mathbf{x}'(t) = A\mathbf{x}(t)$

① Diagonalize $A = PDP^{-1}$

② Solve $\mathbf{y}'(t) = D\mathbf{y}(t)$.

③ Solve $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

$$A = \underbrace{\begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}^{-1}$$

② $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{t} y_1(0) \\ e^{-t} y_2(0) \end{pmatrix}$ free

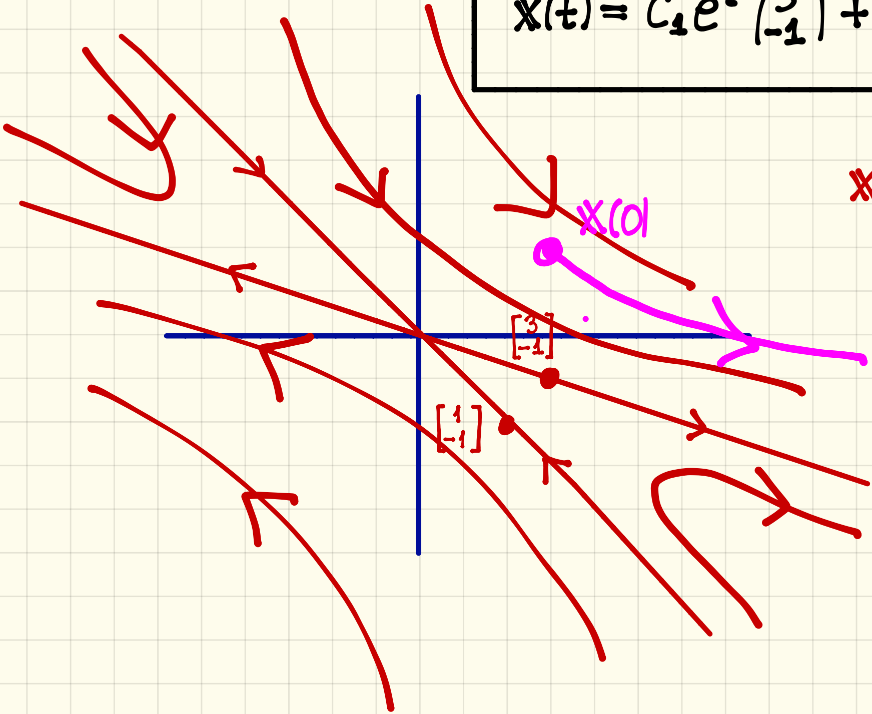
③ Let c_1, c_2 be numbers. $\mathbf{y}(t) = \begin{pmatrix} c_1 e^{t} \\ c_2 e^{-t} \end{pmatrix}$

Solve $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$

$$\Rightarrow P\mathbf{y}(t) = \mathbf{x}(t) \Rightarrow \boxed{\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$\mathbf{x}(t) = c_1 e^{t}\begin{pmatrix}3\\-1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$$

$\mathbf{x}' = A\mathbf{x}$



Solve the initial value problem

$$x_1'(t) = 2x_1(t) + 3x_2(t) \qquad x_1(0) = 3$$

$$x_2'(t) = -x_1(t) - 2x_2(t) \qquad x_2(0) = 2$$

(A) Plug $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ into $\boxed{\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

and solve for c_1 and c_2.

(B) Go back to the solution

$$\mathbf{x}(t) = P\,\mathbf{y}(t)$$

Solving $\mathbf{x}' = A\mathbf{x}$ when A has complex eigenvectors

$$x_1'(t) = a x_1(t) - b x_2(t)$$
$$x_2'(t) = b x_1(t) + a x_2(t)$$

THEOREM: the space of solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ has dimension 2.

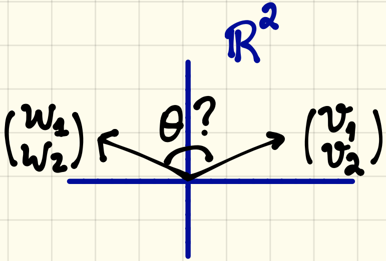
CONSEQUENCE: If $\mathbf{x}'(t) = A\mathbf{x}(t)$, $\mathbf{y}'(t) = A\mathbf{y}(t)$ and $\{\mathbf{x}(t), \mathbf{y}(t)\}$ is LI then ALL solutions are linear combinations of $\mathbf{x}(t)$ and $\mathbf{y}(t)$:

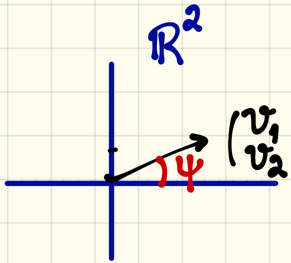
$$c_1 \mathbf{x}(t) + c_2 \mathbf{y}(t).$$

LESSON 30: INNER PRODUCT, LENGHT AND ORTHOGONALITY

Geometry problem:

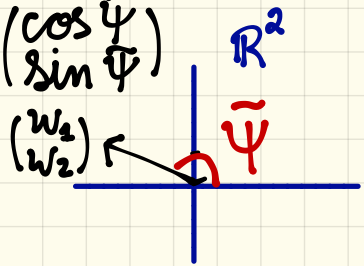
1) Given two vectors in $\mathbb{R}^2$, compute the angle between them.





$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sqrt{v_1^2+v_2^2}\begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$$

$$\sqrt{w_1^2+w_2^2}\begin{pmatrix} \cos\tilde{\psi} \\ \sin\tilde{\psi} \end{pmatrix}$$



$$\theta = \tilde{\psi} - \psi$$

$$\begin{aligned} \cos\theta &= \cos(\tilde{\psi} - \psi) \\ &= \cos\tilde{\psi}\cos\psi + \sin\tilde{\psi}\sin\psi \end{aligned}$$

$$\Rightarrow \sqrt{v_1^2+v_2^2}\sqrt{w_1^2+w_2^2}\cos\theta = v_1 w_1 + v_2 w_2$$

Geometry problem:

1) Given two vectors in $\mathbb{R}^2$, compute the angle between them.

2) Given two vectors in $\mathbb{R}^3$, compute the angle between them.

θ = angle between $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$\Rightarrow \boxed{x_1 y_1 + x_2 y_2 + x_3 y_3 = \|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos\theta}$$

DEFINITION: given $\vec{u} = (u_1 \cdots u_n)^T$ and $\vec{v} = (v_1 \cdots v_n)^T$, the **inner product** between $\vec{u}$ and $\vec{v}$ is the number

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n}$$

NOTE: the inner product and the **dot product** are one and the same thing.

DEFINITION: given $\vec{u} = (u_1 \cdots u_n)^T$ and $\vec{v} = (v_1 \cdots v_n)^T$, the **distance** between $\vec{u}$ and $\vec{v}$ is the number

$$\boxed{\|\vec{u} - \vec{v}\| = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}}$$

It turns out that $\|\vec{u} - \vec{v}\| = (u_1 - v_1)^2 + \cdots + (u_n - v_n)^2$

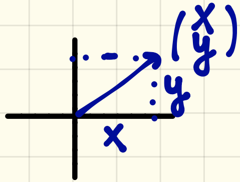
$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n}$$

DEFINITION: given $\vec{u} = (u_1 \cdots u_n)^T$ and $\vec{v} = (v_1 \cdots v_n)^T$, the **distance** between $\vec{u}$ and $\vec{v}$ is the number

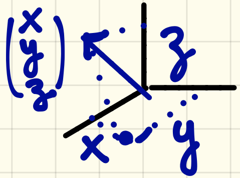
$$\boxed{\|\vec{u} - \vec{v}\| = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}}$$

The **norm** of $\vec{u}$ is the distance between $\vec{u}$ and $\vec{0}$.

$$\boxed{\|u\| = \sqrt{\vec{u} \cdot \vec{u}}}$$



$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \sqrt{x^2 + y^2}$$



$$\left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\| = \sqrt{x^2 + y^2 + z^2}$$

PROPERTIES OF THE INNER PRODUCT (THM 1 PAGE 333)

Ⓐ If $\vec{u} \neq \vec{0}$ then $\vec{u} \cdot \vec{u}$ is POSITIVE.

Ⓑ The function

$$(\vec{u}, \vec{v}) \longmapsto \vec{u} \cdot \vec{v}$$

(domain $\mathbb{R}^n \times \mathbb{R}^n$, codomain $\mathbb{R}$) is BILINEAR.

Ⓒ $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

Ⓐ $\vec{u} \cdot \vec{u} = u_1^2 + \cdots + u_n^2 \geq 0$ and is $= 0$ only if $\vec{u} = \vec{0}$

Ⓑ Bilinear means

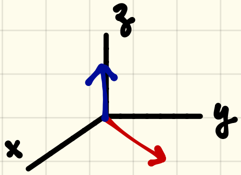
$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w}$$

$$(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$$

DEFINITION: we say that two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal or perpendicular if $\vec{u} \cdot \vec{v} = \vec{0}$.

EXAMPLES:

A) $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$ is orthogonal to $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$.



B) iz is orthogonal to z for any $z = a+ib$

$iz = -b + ia$

$\begin{pmatrix} -b \\ a \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

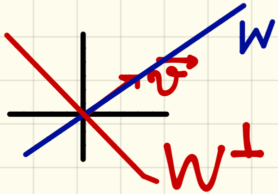
DEFINITION: we say that two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are **orthogonal** or **perpendicular** if $\vec{u} \cdot \vec{v} = \vec{0}$.

DEFINITION: Let W be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of W is the subspace $W^\perp$ of $\mathbb{R}^n$ defined by

$$\vec{u} \in W^\perp \iff \vec{u} \cdot \vec{v} = 0 \text{ for ALL } \vec{v} \text{ in } W.$$

$\mathbb{R}$ if $W = \mathbb{R}$
then $W^\perp = \{0\}$

$\mathbb{R}^2$ if $W = \text{span}\{\vec{v}\}$
then $W^\perp = \text{span}\{\vec{w}\}$ with $\vec{v} \cdot \vec{w} = 0$



THEOREM: Let A be an $m\times n$ matrix

$\vec{u}$ vector in $\mathbb{R}^n$

$\vec{v}$ vector in $\mathbb{R}^m$

$$\Rightarrow \boxed{\underbrace{A}_{m\times n}\underbrace{\vec{u}}_{n\times 1}\cdot\underbrace{\vec{v}}_{m\times 1} = \vec{u}\cdot \underbrace{A^T}_{n\times m}\underbrace{\vec{v}}_{m\times 1}}$$

$n\times 1$

Basic formula: $(AB)^T = B^TA^T$.

Notice that $x_1y_1+x_2y_2+x_3y_3 = [x_1\ \ x_2\ \ x_3]\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix}$

$$\Rightarrow A\vec{u}\cdot\vec{v} = (A\vec{u})^T\vec{v} = u^TA^T\vec{v} = \vec{u}\cdot A^T\vec{v}$$

THEOREM: Let A be an $m \times n$ matrix. Then

$$\boxed{\operatorname{Nul} A = (\operatorname{Col} A^T)^{\perp}}$$

$$\boxed{\operatorname{Col} A = (\operatorname{Nul} A^T)^{\perp}}$$

$$\underset{m \times n}{A}\ \underset{n \times 1}{\vec{v}}$$

$\vec{v} \in \operatorname{Nul} A \iff A\vec{v} = \vec{0}$

$\iff A\vec{v} \cdot \vec{w} = 0$ for ANY $\vec{w}$ in $\mathbb{R}^m$

$\iff \vec{v} \cdot A^T\vec{w} = 0$ for ANY $\vec{w}$ in $\mathbb{R}^m$

$\iff \vec{v}$ is orthogonal to all vectors in $\operatorname{Col} A^T$ $A^T\vec{w}$

$\iff \vec{v} \in (\operatorname{Col} A^T)^{\perp}$

APPENDIX: If A is a $m \times n$ matrix

B is a $n \times p$ matrix

then $\boxed{(AB)^T = B^T A^T}$

$$(AB)^T_{ij} = (AB)_{ji}$$

$$= \Sigma_k A_{jk} B_{ki}$$

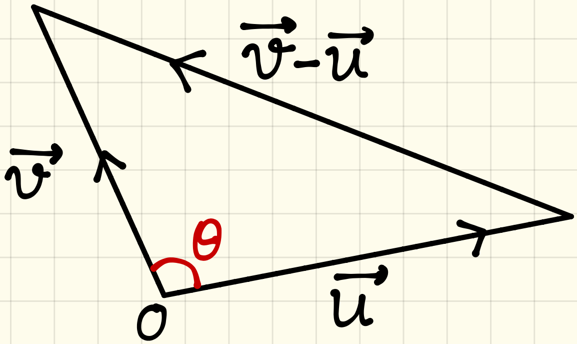
$$= \Sigma_k B^T_{ik} A^T_{kj}$$

$$= (B^T A^T)_{ij}$$

APPENDIX: the formula

$$\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta}$$ in 3 dimensions.

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$



Law of cosines

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$$

$$(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2 = u_1^2+u_2^2+u_3^2+v_1^2+v_2^2+v_3^2 + {}$$
$$-2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$$

$$\Rightarrow -2u_1v_1 - 2u_2v_2 - 2u_3v_3 = -2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$$

DEFINITION: A set $\{\vec{u}_1, \dots, \vec{u}_n\}$ in $\mathbb{R}^m$ is called orthogonal if $\vec{u}_i \cdot \vec{u}_j = 0$ whenever $i \neq j$.

The set is orthonormal if it is orthogonal and all of its vectors have lenght 1.

PROPOSITION: if a $m \times n$ matrix U has orthonormal columns then $U^T U = I$.

$$\vec{u_i} \cdot \vec{u_j} = 0$$

$$\underbrace{\begin{bmatrix} — & \vec{u_1} & — \\ — & \vec{u_2} & — \\ & \vdots & \\ — & \vec{u_n} & — \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} | & | & & | \\ \vec{u_1} & \vec{u_2} & \cdots & \vec{u_n} \\ | & | & & | \end{bmatrix}}_{U} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{bmatrix}$$

entry $(i,j) = \vec{u_i} \cdot \vec{u_j}$

PROPOSITION: if a $m\times n$ matrix U has orthonormal columns then $U^TU=I$.

$$\begin{bmatrix} — & \vec{u}_1 & — \\ — & \vec{u}_2 & — \\ & \vdots & \\ — & \vec{u}_n & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

CONSEQUENCE 1: orthonormal sets are LI.

LI means that if $x_1\vec{u}_1+\cdots+x_n\vec{u}_n=\vec{0}$ then $x_1=\cdots=x_n=0$.

Assume $U\vec{x}=\vec{0}$, let's show $\vec{x}=\vec{0}$.

$U\vec{x}=\vec{0} \Rightarrow U^TU\vec{x}=U^T\vec{0} \Rightarrow \vec{x}=\vec{0}.$

PROPOSITION: if a $m \times n$ matrix U has orthonormal columns then $U^T U = I$.

$$\begin{bmatrix} — & \vec{u}_1 & — \\ — & \vec{u}_2 & — \\ & \vdots & \\ — & \vec{u}_n & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

CONSEQUENCE 1: orthonormal sets are LI.

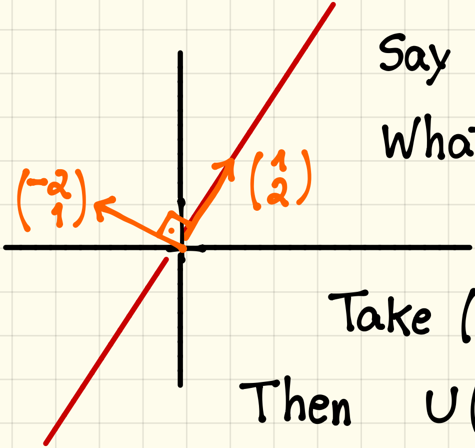
CONSEQUENCE 2: $U\vec{x} \cdot U\vec{y} = \vec{x} \cdot \vec{y}$ for any $\vec{x}, \vec{y}$ in $\mathbb{R}^n$.

In other words U preserves angles and lengths.

$$U\vec{x} \cdot U\vec{y} = \vec{x} \cdot U^T U \vec{y} = \vec{x} \cdot I \vec{y} = \vec{x} \cdot \vec{y}$$

EXAMPLE OF MATRIX WITH ORTHONORMAL COLUMNS

Reflection through a line through the origin in $\mathbb{R}^2$.



Say U = reflection through $\text{span}\{\binom{1}{2}\}$

What is the matrix representation of U?

We need the image of two LI vectors.

Take $\binom{1}{2}$ and $\binom{-2}{1}$, which is orthogonal to $\binom{1}{2}$.

Then $U\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$. Solving for U, we find

$$\boxed{U = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}}$$

Check that the columns of U form an orthonormal set.

PROBLEM: Given an orthonormal basis $\{\vec{u}_1, \cdots, \vec{u}_n\}$ of $\mathbb{R}^n$

a vector $\vec{u}$ in $\mathbb{R}^n$

find the coordinates of $\vec{u}$ in the basis $\{\vec{u}_1, \cdots, \vec{u}_n\}$.

We look for $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{pmatrix}$ such that $U\vec{x} = \vec{u}$.

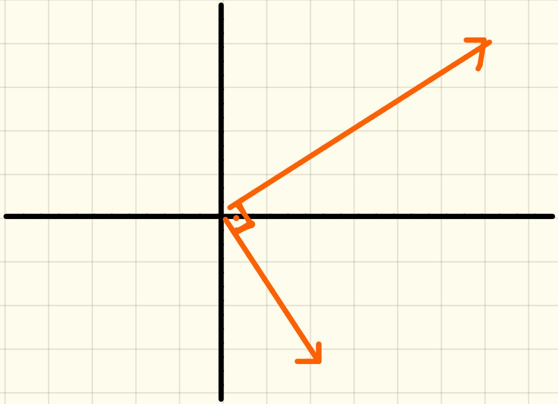
$$U = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix}$$

Since the columns of U are orthonormal, $U^T U = I$

$$\Rightarrow U^T U \vec{x} = U^T \vec{u}$$

$$\Rightarrow \boxed{\vec{x} = U^T \vec{u}}$$

EXAMPLE: express $\begin{bmatrix}9\\-7\end{bmatrix}$ as a linear combination of $\begin{bmatrix}2\\-3\end{bmatrix}$ and $\begin{bmatrix}6\\4\end{bmatrix}$.



We look for $\begin{bmatrix}x\\y\end{bmatrix}$ such

$$\begin{bmatrix}2 & 6\\-3 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}9\\-7\end{bmatrix}$$

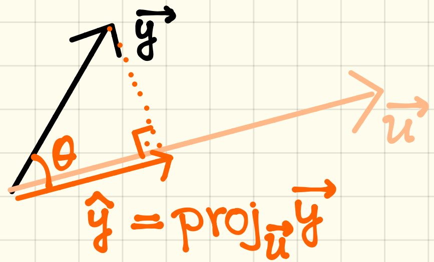
$$\begin{bmatrix}2 & -3\\6 & 4\end{bmatrix}\begin{bmatrix}2 & 6\\-3 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2 & -3\\6 & 4\end{bmatrix}\begin{bmatrix}9\\-7\end{bmatrix}$$

$$\begin{bmatrix}13 & 0\\0 & 52\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}39\\26\end{bmatrix}$$

$$\Rightarrow \boxed{\begin{matrix}x = 3\\ y = \frac{1}{2}\end{matrix}}$$

Check! $\begin{bmatrix}2\\-3\end{bmatrix}3 + \begin{bmatrix}6\\4\end{bmatrix}\frac{1}{2} \stackrel{?}{=} \begin{bmatrix}9\\-7\end{bmatrix}$

PROBLEM: given vectors $\vec{y}$ and $\vec{u}$ in $\mathbb{R}^n$, find a formula for the projection of $\vec{y}$ onto $\operatorname{span}\{\vec{u}\}$.



$$\hat{y} = \underbrace{\frac{\vec{u}}{|\vec{u}|}}_{\text{points in direction } \vec{u} \text{ has lenght } 1} \cdot |\hat{y}| = \frac{\vec{u}}{|\vec{u}|} \cdot |\vec{y}| \cdot \cos\theta$$

But

$$\vec{y} \cdot \vec{u} = |\vec{y}| \cdot |\vec{u}| \cdot \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{y} \cdot \vec{u}}{|\vec{y}| \cdot |\vec{u}|}$$

$$\Rightarrow \hat{y} = \frac{\vec{u}}{|\vec{u}|} \cdot |\vec{y}| \cdot \frac{\vec{y} \cdot \vec{u}}{|\vec{y}| \cdot |\vec{u}|} = \underbrace{\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}}_{\text{number}} \cdot \underbrace{\vec{u}}_{\text{vector}}$$

$$\Rightarrow \boxed{\hat{y} = \operatorname{Proj}_{\vec{u}} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}}$$

LESSON 32: ORTHOGONAL PROJECTIONS

DEFINITION: Given a vector $\vec{y}$ in $\mathbb{R}^n$ and a subspace W of $\mathbb{R}^n$, the **orthogonal projection** of $\vec{y}$ onto W is the unique vector $\hat{y}$ in W such that $\vec{y}-\hat{y}$ is in $W^\perp$.

A FORMULA FOR THE ORTHOGONAL PROJECTION

Let W be a subspace of $\mathbb{R}^n$ and $\vec{y}$ a vector in $\mathbb{R}^n$.

If $\{\vec{u_1}, \dots, \vec{u_p}\}$ is an ORTHOGONAL basis of W

then

$$\boxed{\operatorname{proj}_W \vec{y} = \frac{(\vec{y} \cdot \vec{u_1})}{(\vec{u_1} \cdot \vec{u_1})} \vec{u_1} + \cdots + \frac{(\vec{y} \cdot \vec{u_p})}{(\vec{u_p} \cdot \vec{u_p})} \vec{u_p}}$$

Let $\hat{y} = \operatorname{proj}_W \vec{y}$.

① $\hat{y}$ is in W

② $y - \hat{y}$ is in $W^\perp$

because $(y - \hat{y}) \cdot \vec{u_i} = 0$

ORTHOGONAL DECOMPOSITION

If W is a subspace of $\mathbb{R}^n$ then any vector $\vec{y}$ in $\mathbb{R}^n$ can be written as $\boxed{\vec{y} = \underset{W}{\hat{y}} + \underset{W^\perp}{y^\perp}}$ in only one way.

If $\vec{y} = \underset{W}{\hat{y}} + \underset{W^\perp}{y^\perp} = \underset{W}{\tilde{y}} + \underset{W^\perp}{\tilde{y}^\perp}$ then $\underset{W}{\hat{y} - \tilde{y}} = \underset{W^\perp}{\tilde{y}^\perp - y^\perp}$

$$\Rightarrow (\hat{y}-\tilde{y}) \bullet (\hat{y}-\tilde{y}) = \underset{W^\perp}{(\tilde{y}^\perp - y^\perp)} \bullet \underset{W}{(\hat{y}-\tilde{y})} = 0$$

$$\Rightarrow \hat{y} = \tilde{y} \text{ and } y^\perp = \tilde{y}^\perp.$$

ORTHOGONAL DECOMPOSITION

If W is a subspace of $\mathbb{R}^n$ then any vector $\vec{y}$ in $\mathbb{R}^n$ can be written as $\boxed{\vec{y} = \underbrace{\hat{y}}_{W} + \underbrace{y^{\perp}}_{W^{\perp}}}$ in only one way.

CONSEQUENCE: if $\{\vec{u}_1, \dots, \vec{u}_p\}$ and $\{\vec{v}_1, \dots, \vec{v}_p\}$ are ORTHONORMAL basis of W then

$$\hat{y} = (\vec{y}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{y}\cdot\vec{u}_p)\vec{u}_p = (\vec{y}\cdot\vec{v}_1)\vec{v}_1 + \cdots + (\vec{y}\cdot\vec{v}_p)\cdot\vec{v}_p.$$

PROPOSITION: the orthogonal projection of $\vec{y}$ onto W is the vector in W that is closest to $\vec{y}$.

We know $\vec{y} = \underbrace{\hat{y}}_{W} + \underbrace{y^{\perp}}_{W^{\perp}}$.

If $\vec{w}$ is in W, we claim $\|\vec{y} - \vec{w}\| \geq \|y^{\perp}\|$.

$$\vec{y} - \vec{w} = \underbrace{\hat{y} - \vec{w}}_{W} + \underbrace{y^{\perp}}_{W^{\perp}} \Rightarrow \|\vec{y} - \vec{w}\|^2 = \underbrace{\|\hat{y} - \vec{w}\|^2}_{\geq 0} + \|y^{\perp}\|^2$$

$$\geq \|y^{\perp}\|^2.$$

SUMMARY

•) If W is a subspace of $\mathbb{R}^n$ then any vector $\vec{y}$ in $\mathbb{R}^n$ can be written as $\boxed{\vec{y} = \underbrace{\hat{y}}_{W} + \underbrace{y^{\perp}}_{W^{\perp}}}$ in a unique way.

••) $\hat{y}$ is the point of W that is closest to $\vec{y}$.

∴) If $\{\vec{u_1}, \dots, \vec{u_p}\}$ is an orthonormal basis of W then

$$\boxed{\hat{y} = (\vec{y} \cdot \vec{u_1})\vec{u_1} + \cdots + (\vec{y} \cdot \vec{u_p})\vec{u_p}}$$

EXERCISE: $\vec{u_1} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ $\vec{u_2} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ Find a vector $\vec{v}$ that is orthogonal to both $\vec{u_1}$ and $\vec{u_2}$.

GIVEN: $\vec{u_1} \cdot \vec{u_2} = 0$ and $\vec{u_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is not in span$\{\vec{u_1}, \vec{u_2}\}$.

If we denote $W = \text{span}\{\vec{u_1}, \vec{u_2}\}$ and decompose

$$\vec{u_3} = \underbrace{\hat{u}_3}_{W} + \underbrace{u_3^{\perp}}_{W^{\perp}}$$

then $u_3^{\perp}$ is orthogonal to both $\vec{u_1}$ and $\vec{u_2}$.

We have a formula for $\hat{u}_3$:

$$\hat{u}_3 = \frac{\vec{u_3} \cdot \vec{u_1}}{\vec{u_1} \cdot \vec{u_1}} \vec{u_1} + \frac{\vec{u_3} \cdot \vec{u_2}}{\vec{u_2} \cdot \vec{u_2}} \vec{u_2} = \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{-1}{30} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{5} \\ -\frac{2}{5} \end{bmatrix}$$

Now $u_3^{\perp} = \vec{u_3} - \hat{u}_3 = \begin{bmatrix} 0 \\ \frac{4}{5} \\ \frac{2}{5} \end{bmatrix}$.

EXERCISE: Let $\vec{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$ $\vec{u_1} = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$ $\vec{u_2} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$. Find the distance from $\vec{y}$ to the plane spanned by $\vec{u_1}$ and $\vec{u_2}$.

"Luckily" $\vec{u_1}$ and $\vec{u_2}$ are orthogonal.

If $\vec{y} = \hat{y} + y^{\perp}$ with $\hat{y}$ in $\operatorname{span}\{\vec{u_1}, \vec{u_2}\}$ and $y^{\perp}$ in $\operatorname{span}\{u_1, u_2\}^{\perp}$

then the distance from $\vec{y}$ to $\operatorname{span}\{\vec{u_1}, \vec{u_2}\}$ is $\|y^{\perp}\|$.

We know how to compute $\hat{y}$:

$$\hat{y} = \frac{\vec{y}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}}\vec{u_1} + \frac{\vec{y}\cdot\vec{u_2}}{\vec{u_2}\cdot\vec{u_2}}\vec{u_2}$$

$$= \frac{35}{35}\begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} + \frac{(-28)}{14}\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}$$

$$\Rightarrow y^{\perp} = y - \hat{y} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

Answer: the distance from $\vec{y}$ to $\operatorname{span}\{\vec{u_1}, \vec{u_2}\}$ is

$\|y^{\perp}\| = \sqrt{2^2+0^2+6^2} = 2\sqrt{10}$.

THE ALGORITHM

INPUT: a LI sequence $(\vec{u}_1, \dots, \vec{u}_m)$ in $\mathbb{R}^n$

OUTPUT: an orthogonal sequence $(\vec{q}_1, \dots, \vec{q}_m)$ in $\mathbb{R}^n$

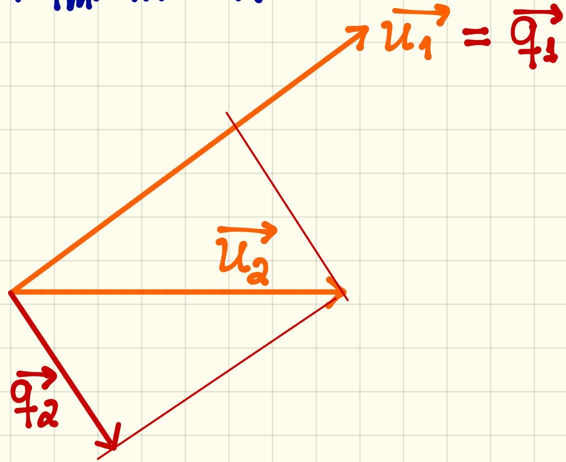
$$\vec{q}_1 := \vec{u}_1$$

$$\vec{q}_2 := \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \cdot \vec{q}_1$$

$$\vec{q}_3 := \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \cdot \vec{q}_1 - \frac{\vec{u}_3 \cdot \vec{q}_2}{\vec{q}_2 \cdot \vec{q}_2} \cdot \vec{q}_2$$

...

$$\vec{q}_m := \vec{u}_m - \frac{\vec{u}_m \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \cdot \vec{q}_1 - \frac{\vec{u}_m \cdot \vec{q}_2}{\vec{q}_2 \cdot \vec{q}_2} \cdot \vec{q}_2 \cdots - \frac{\vec{u}_m \cdot \vec{q}_{m-1}}{\vec{q}_{m-1} \cdot \vec{q}_{m-1}} \cdot \vec{q}_{m-1}$$



EXAMPLE: Find an orthogonal basis for the column space of

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

Let $\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \vec{q}_1$ $\quad \vec{u}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}$ and $\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}$.

Then

$$\vec{q}_2 := \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \cdot \vec{q}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \frac{(-40)}{20} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{q}_3 := \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \cdot \vec{q}_1 - \frac{\vec{u}_3 \cdot \vec{q}_2}{\vec{q}_2 \cdot \vec{q}_2} \cdot \vec{q}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{30}{20} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{(-10)}{20} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

EXAMPLE: Find an orthogonal basis for the column space of

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

A possible orthogonal basis is

$$\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}.$$

WHAT HAPPENS IN THE ALGORITHM IF THE INPUT VECTORS ARE LD?

Some of the $\vec{q_j}$ will be zero.

Say $\{\vec{u_1}, \vec{u_2}\}$ is LI but $\vec{u_3}$ is in span $\{\vec{u_1}, \vec{u_2}\}$.

Then $\{\vec{q_1}, \vec{q_2}\}$ is orthogonal and $\vec{q_3} = 0$.

In any case, span $\{\vec{u_1}, \ldots, \vec{u_j}\} =$ span $\{\vec{q_1}, \ldots, \vec{q_j}\}$.

THE QR FACTORIZATION

Matrix version of the Gram-Schmidt algorithm.

Let $U = \begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \vec{u_3} \\ | & | & | \end{bmatrix}$

$$\vec{q_1} := \vec{u_1}$$

$$\vec{q_2} := \vec{u_2} - \frac{\vec{u_2}\cdot\vec{q_1}}{\vec{q_1}\cdot\vec{q_1}}\cdot\vec{q_1}$$

$$\vec{q_3} := \vec{u_3} - \frac{\vec{u_3}\cdot\vec{q_1}}{\vec{q_1}\cdot\vec{q_1}}\cdot\vec{q_1} - \frac{\vec{u_3}\cdot\vec{q_2}}{\vec{q_2}\cdot\vec{q_2}}\cdot\vec{q_2}$$

$$\begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \vec{u_3} \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{q_1} & \vec{q_2} & \vec{q_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix}$$

THE QR FACTORIZATION

$$\vec{q}_1 := \vec{u}_1$$

$$\vec{q}_2 := \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \cdot \vec{q}_1$$

$$\vec{q}_3 := \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \cdot \vec{q}_1 - \frac{\vec{u}_3 \cdot \vec{q}_2}{\vec{q}_2 \cdot \vec{q}_2} \cdot \vec{q}_2$$

$$\vec{u}_1 = \vec{q}_1$$

$$\vec{u}_2 = \frac{\vec{u}_2 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \cdot \vec{q}_1 + \vec{q}_2$$

$$\vec{u}_3 = \frac{\vec{u}_3 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \cdot \vec{q}_1 + \frac{\vec{u}_3 \cdot \vec{q}_2}{\vec{q}_2 \cdot \vec{q}_2} \cdot \vec{q}_2 + \vec{q}_3$$

$$\begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 & \frac{\vec{u}_2 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} & \frac{\vec{u}_3 \cdot \vec{q}_1}{\vec{q}_1 \cdot \vec{q}_1} \\ 0 & 1 & \frac{\vec{u}_3 \cdot \vec{q}_2}{\vec{q}_2 \cdot \vec{q}_2} \\ 0 & 0 & 1 \end{bmatrix}$$

THE QR FACTORIZATION

A matrix U whose columns are LI can always be factored as $U = QR$ where

Q has orthonormal columns $\leftarrow Q^TQ = I$

R is upper triangular

EXAMPLE: Find a QR factorization for the matrix

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

We have computed

$$\vec{q}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\vec{q}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

Whence

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The columns of the left matrix are not orthonormal. We need to normalize.

EXAMPLE: Find a QR factorization for the matrix

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The columns of the left matrix are not orthonormal. We need to normalize.

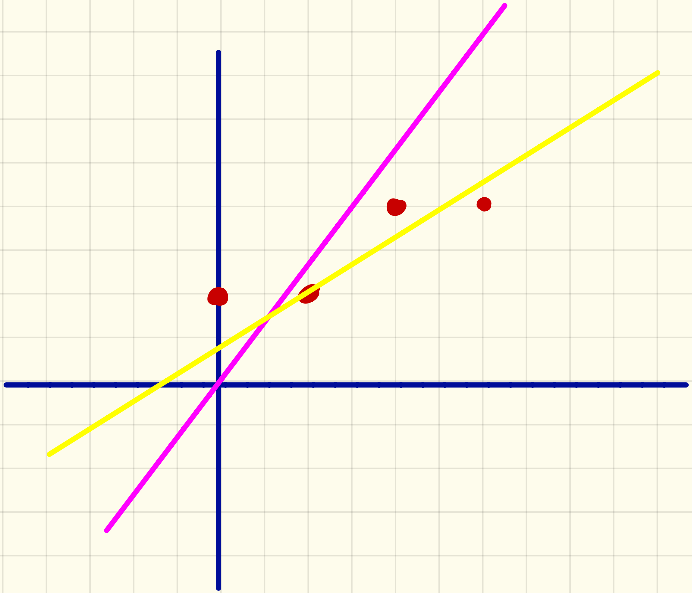
It turns out that the three columns have norm $\sqrt{20}$.

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} = \underbrace{\begin{bmatrix} 3/\sqrt{20} & 1/\sqrt{20} & -3/\sqrt{20} \\ 1/\sqrt{20} & 3/\sqrt{20} & 1/\sqrt{20} \\ -1/\sqrt{20} & 3/\sqrt{20} & 1/\sqrt{20} \\ 3/\sqrt{20} & -1/\sqrt{20} & 3/\sqrt{20} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{20} & -2\sqrt{20} & 3\sqrt{20}/2 \\ 0 & \sqrt{20} & -\sqrt{20}/2 \\ 0 & 0 & \sqrt{20} \end{bmatrix}}_{R}$$

LESSON 34: LEAST-SQUARES PROBLEMS (SECTION 6.5)

PROBLEM: find the line that best fits the points

$(0,1), (1,1), (2,2), (3,2)$.



What is "best fit"?

We look for β_0 and β_1 such that

approximately

$$\beta_0 + \beta_1 0 \approx 1$$
$$\beta_0 + \beta_1 1 \approx 1$$
$$\beta_0 + \beta_1 2 \approx 2$$
$$\beta_0 + \beta_1 3 \approx 2$$

PROBLEM: find the line that best fits the points

$(0,1), (1,1), (2,2), (3,2)$.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{X} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \approx \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}}_{y}$$

What is "best fit"?

We look for β_0 and β_1 such that

approximately

$$\beta_0 + \beta_1 0 \approx 1$$

$$\beta_0 + \beta_1 1 \approx 1$$

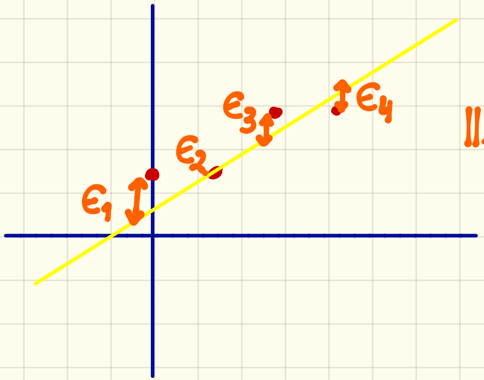
$$\beta_0 + \beta_1 2 \approx 2$$

$$\beta_0 + \beta_1 3 \approx 2$$

unknown

Approximate solution for $X\beta = y$

$\|X\beta - y\|$ minimal among all β

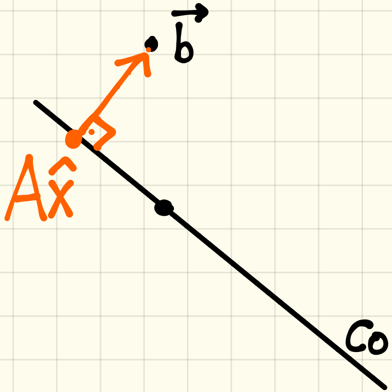


$$\|X\beta - y\| = \sqrt{\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2}$$

DEFINITION: Let A be an $m\times n$ matrix and $\vec{b}$ a vector in $\mathbb{R}^m$. We say that a vector $\hat{x}$ in $\mathbb{R}^n$ is a **least-squares solution of $A\vec{x}=\vec{b}$** if $\|\vec{b}-A\hat{x}\| \leq \|\vec{b}-A\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$.

The number $\|\vec{b}-A\hat{x}\|$ is called the **least-squares error** of $\hat{x}$.

THEOREM Let A be an $m \times n$ matrix and $\vec{b}$ a vector in $\mathbb{R}^m$. A vector $\hat{x}$ in $\mathbb{R}^n$ is a least-squares solution of $Ax = \vec{b}$ if and only if $\boxed{A^T A \hat{x} = A^T \vec{b}}$.



Want $A\hat{x} = \mathrm{proj}_{\mathrm{Col}\,A} \vec{b}$

$\Rightarrow \vec{b} - A\hat{x}$ is orthogonal to Col A

$\Rightarrow A^T(\vec{b} - A\hat{x}) = 0 \Rightarrow A^T\vec{b} = A^T A \hat{x}$.

Col A = the set of all $A\vec{x}$

WHEN IS THE LEAST-SQUARES SOLUTION UNIQUE?

THEOREM: if the columns of A are LI then $A^TA\hat{x}=A^Tb$ has only one solution.

If the columns of A are LD the $A^TA\hat{x}=A^Tb$ can have several solutions.

↪ columns of A LI $\Rightarrow$ if $A^TAx=A^Tb$ and $A^TAy=A^Tb$ then $A^TA(x-y)\cdot(x-y)=0$

$$\Rightarrow \|A(x-y)\|^2=0$$

$$\Rightarrow A(x-y)=0$$

$$\xRightarrow{\text{LI columns}} x-y=0$$

↪ columns of A LD $\Rightarrow$ there is $\vec{v}\neq 0$ such that $A\vec{v}=0$

if $A^TA\hat{x}=A^Tb$

then $A^TA(\hat{x}+t\vec{v})=A^Tb$ for any t in $\mathbb{R}$

EXAMPLE: Find a least-squares solution of $Ax=b$ where

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix} \qquad b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

EXAMPLE: Find a least-squares solution of $Ax=b$ where

$$A=\begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix} \qquad b=\begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A^TA\hat{x}=A^Tb$$

$$A^T=\begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix} \quad A^TA=\begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} \quad A^Tb=\begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix}$$

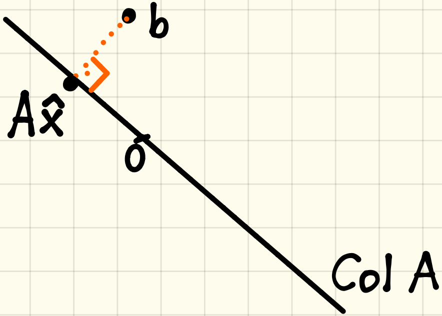
$$\begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

EXAMPLE: Find a least-squares solution of $Ax=b$ where

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix} \qquad b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2nd solution: exploit the orthogonality of the columns of A. Instead of using $A^TAx = A^Tb$, go back to the original system $Ax=b$.

$$A\hat{x} = \text{proj}_{\text{Col}(A)}\, b$$

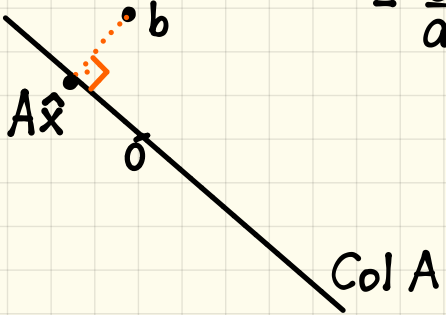


$$= \frac{b\cdot a_1}{a_1\cdot a_1} a_1 + \frac{b\cdot a_2}{a_2\cdot a_2} a_2 + \frac{b\cdot a_3}{a_3\cdot a_3} a_3$$

EXAMPLE: Find a least-squares solution of $Ax=b$ where

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix} \qquad b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A\hat{x} = \text{proj}_{\text{Col}(A)}\, b$$



$$= \frac{b \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{b \cdot a_2}{a_2 \cdot a_2} a_2 + \frac{b \cdot a_3}{a_3 \cdot a_3} a_3$$

$$= \frac{36}{54} \begin{bmatrix} 4 \\ 1 \\ 6 \\ 1 \end{bmatrix} + \frac{0}{27} \begin{bmatrix} 0 \\ -5 \\ 1 \\ -1 \end{bmatrix} + \frac{9}{27} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -5 \end{bmatrix}$$

EXAMPLE: Find a least-squares solution of $Ax=b$ where

$$A=\begin{bmatrix}4&0&1\\1&-5&1\\6&1&0\\1&-1&-5\end{bmatrix}\qquad b=\begin{bmatrix}9\\0\\0\\0\end{bmatrix}$$

$$A\hat{x}=\frac{36}{54}\begin{bmatrix}4\\1\\6\\1\end{bmatrix}-\frac{0}{27}\begin{bmatrix}0\\-5\\1\\-1\end{bmatrix}-\frac{9}{27}\begin{bmatrix}1\\1\\0\\-5\end{bmatrix}$$

$$\begin{bmatrix}4&0&1\\1&-5&1\\6&1&0\\1&-1&-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{36}{54}\begin{bmatrix}4\\1\\6\\1\end{bmatrix}+\frac{0}{27}\begin{bmatrix}0\\-5\\1\\-1\end{bmatrix}+\frac{9}{27}\begin{bmatrix}1\\1\\0\\-5\end{bmatrix}$$

$$\Rightarrow\quad\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2/3\\0\\1/3\end{bmatrix}$$

PROBLEM: find the line that best fits the points

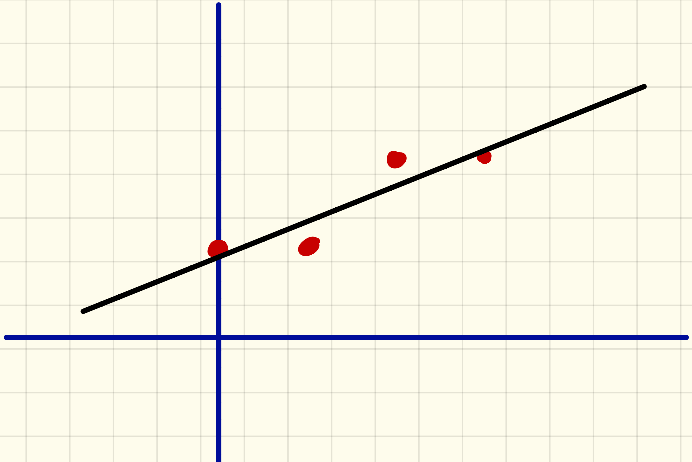
(0,1), (1,1), (2,2), (3,2).

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{X} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \approx \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}}_{y}$$

We look for $\hat{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ such that

$$X^T X \hat{\beta} = X^T y$$

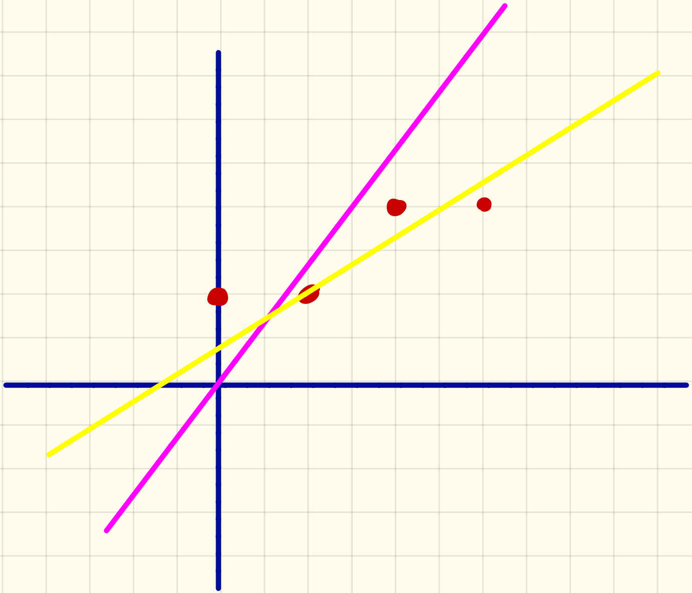
$$\underbrace{\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}}_{X^T X} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}}_{\hat{\beta}} = \underbrace{\begin{bmatrix} 6 \\ 11 \end{bmatrix}}_{y} \Rightarrow \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} .9 \\ .4 \end{bmatrix}$$



LESSON 34: LEAST-SQUARES PROBLEMS (SECTION 6.5)

PROBLEM: find the line that best fits the points

$(0,1), (1,1), (2,2), (3,2)$.



What is "best fit"?

We look for β_0 and β_1 such that

approximately

$$\beta_0 + \beta_1 0 \approx 1$$
$$\beta_0 + \beta_1 1 \approx 1$$
$$\beta_0 + \beta_1 2 \approx 2$$
$$\beta_0 + \beta_1 3 \approx 2$$

PROBLEM: find the line that best fits the points

$(0,1), (1,1), (2,2), (3,2)$.

What is "best fit"?

We look for β_0 and β_1 such that

approximately

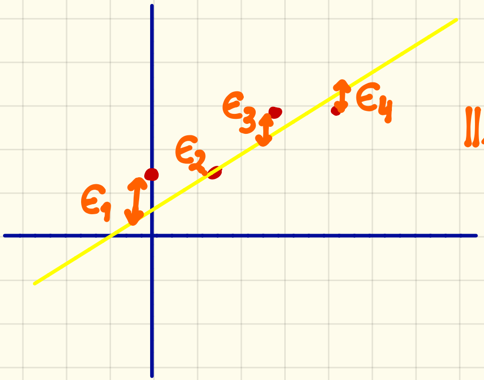
$$\beta_0 + \beta_1 0 \approx 1$$
$$\beta_0 + \beta_1 1 \approx 1$$
$$\beta_0 + \beta_1 2 \approx 2$$
$$\beta_0 + \beta_1 3 \approx 2$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{X} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \approx \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}}_{y}$$

unknown

Approximate solution for $X\beta = y$

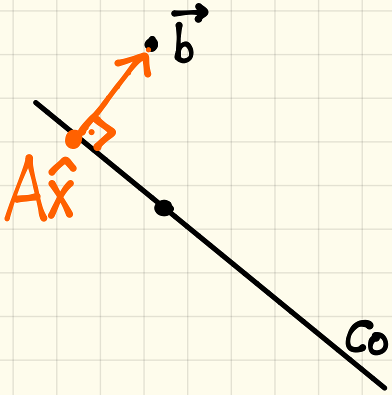
$\|X\beta - y\|$ minimal among all β



$$\|X\beta - y\| = \sqrt{\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2}$$

DEFINITION: Let A be an $m \times n$ matrix and $\vec{b}$ a vector in $\mathbb{R}^m$. We say that a vector $\hat{x}$ in $\mathbb{R}^n$ is a **least-squares solution of $A\vec{x}=\vec{b}$** if $\|\vec{b}-A\hat{x}\| \leq \|\vec{b}-A\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$. The number $\|\vec{b}-A\hat{x}\|$ is called the **least-squares error** of $\hat{x}$.

THEOREM Let A be an $m \times n$ matrix and $\vec{b}$ a vector in $\mathbb{R}^m$. A vector $\hat{x}$ in $\mathbb{R}^n$ is a least-squares solution of $A x = \vec{b}$ if and only if $\boxed{A^T A \hat{x} = A^T \vec{b}}$.



Want $A\hat{x} = \text{proj}_{\text{Col}\,A} \vec{b}$

$\Rightarrow \vec{b} - A\hat{x}$ is orthogonal to Col A

$\Rightarrow A^T(\vec{b} - A\hat{x}) = 0 \Rightarrow A^T\vec{b} = A^T A \hat{x}$.

Col A = the set of all $A\vec{x}$

WHEN IS THE LEAST-SQUARES SOLUTION UNIQUE?

THEOREM: if the columns of A are LI then $A^TA\hat{x}=A^Tb$ has only one solution.

If the columns of A are LD the $A^TA\hat{x}=A^Tb$ can have several solutions.

↪ columns of A LI $\Rightarrow$ if $A^TAx=A^Tb$ and $A^TAy=A^Tb$ then $A^TA(x-y)\cdot(x-y)=0$

$$\Rightarrow \|A(x-y)\|^2=0$$

$$\Rightarrow A(x-y)=0$$

LI columns

$$\Rightarrow x-y=0$$

↪ columns of A LD $\Rightarrow$ there is $\vec{v}\neq 0$ such that $A\vec{v}=0$

if $A^TA\hat{x}=A^Tb$

then $A^TA(\hat{x}+t\vec{v})=A^Tb$ for any t in $\mathbb{R}$

EXAMPLE: Find a least-squares solution of $Ax=b$ where

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix} \qquad b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A^T A \hat{x} = A^T b$$

$$A^T = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix} \quad A^T A = \begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} \quad A^T b = \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

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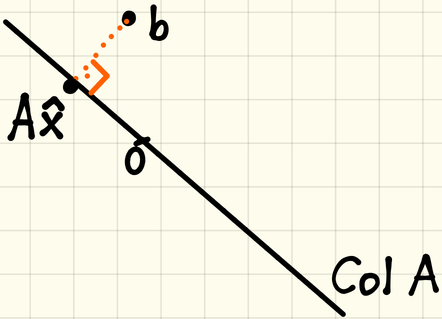
2nd solution: exploit the orthogonality of the columns of A. Instead of using $A^TAx=A^Tb$, go back to the original system $Ax=b$.

$$A\hat{x} = \text{proj}_{\text{Col}(A)}\, b$$



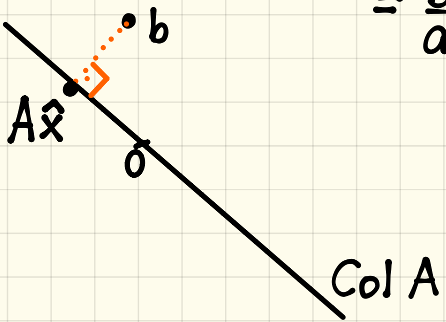
$$= b - \frac{b\cdot a_1}{a_1\cdot a_1}a_1 - \frac{b\cdot a_2}{a_2\cdot a_2}a_2 - \frac{b\cdot a_3}{a_3\cdot a_3}a_3$$

EXAMPLE: Find a least-squares solution of $Ax=b$ where

$$A=\begin{bmatrix}4&0&1\\1&-5&1\\6&1&0\\1&-1&-5\end{bmatrix}\qquad b=\begin{bmatrix}9\\0\\0\\0\end{bmatrix}$$

$$A\hat{x}=\text{proj}_{\text{Col}(A)}\, b$$

$$=\frac{b\cdot a_1}{a_1\cdot a_1}a_1+\frac{b\cdot a_2}{a_2\cdot a_2}a_2+\frac{b\cdot a_3}{a_3\cdot a_3}a_3$$



$$=\frac{36}{54}\begin{bmatrix}4\\1\\6\\1\end{bmatrix}+\frac{0}{27}\begin{bmatrix}0\\-5\\1\\-1\end{bmatrix}+\frac{9}{27}\begin{bmatrix}1\\1\\0\\-5\end{bmatrix}$$

EXAMPLE: Find a least-squares solution of $Ax=b$ where

$$A=\begin{bmatrix}4&0&1\\1&-5&1\\6&1&0\\1&-1&-5\end{bmatrix}\qquad b=\begin{bmatrix}9\\0\\0\\0\end{bmatrix}$$

$$A\hat{x}=\frac{36}{54}\begin{bmatrix}4\\1\\6\\1\end{bmatrix}-\frac{0}{27}\begin{bmatrix}0\\-5\\1\\-1\end{bmatrix}-\frac{9}{27}\begin{bmatrix}1\\1\\0\\-5\end{bmatrix}$$

$$\begin{bmatrix}4&0&1\\1&-5&1\\6&1&0\\1&-1&-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{36}{54}\begin{bmatrix}4\\1\\6\\1\end{bmatrix}+\frac{0}{27}\begin{bmatrix}0\\-5\\1\\-1\end{bmatrix}+\frac{9}{27}\begin{bmatrix}1\\1\\0\\-5\end{bmatrix}$$

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$(0,1), (1,1), (2,2), (3,2).$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{X} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

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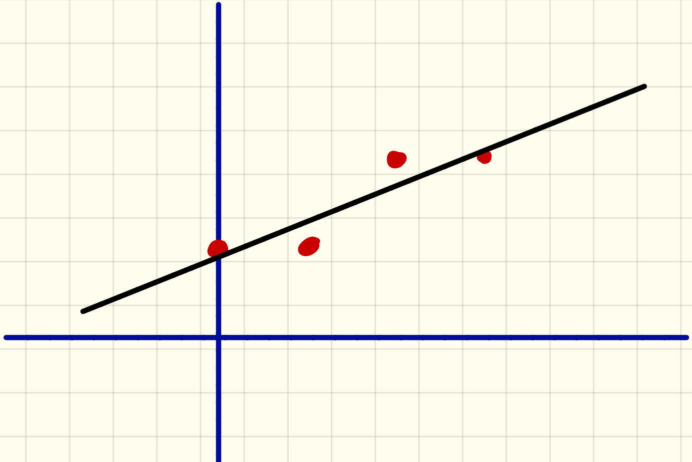
$$X^T X \hat{\beta} = X^T y$$

$$\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} .9 \\ .4 \end{bmatrix}$$



DEFINITION: Let V be a vector space. An **inner product** on V is a function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the axioms:

i) $\langle \cdot , \cdot \rangle$ is bilinear and symmetric.

ii) If $\vec{v} \neq \vec{0}$ then $\langle \vec{v}, \vec{v} \rangle > 0$.

DEFINITION: the **norm** associated to the inner product $\langle \cdot , \cdot \rangle$ is the function $\| \cdot \| : V \to \mathbb{R}$ defined by $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.

DEFINITION: Let V be a vector space. An inner product on V is a function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the axioms:

i) $\langle \cdot , \cdot \rangle$ is bilinear and symmetric.

ii) If $\vec{v} \neq \vec{0}$ then $\langle \vec{v}, \vec{v} \rangle > 0$.

FIRST CONSEQUENCE OF THE AXIOMS

iii) $\langle \vec{0}, \vec{0} \rangle = 0$

$$\cancel{\langle \vec{0}, \vec{0} \rangle} = \langle \vec{0} + \vec{0}, \vec{0} \rangle \overset{i}{=} \cancel{\langle \vec{0}, \vec{0} \rangle} + \langle \vec{0}, \vec{0} \rangle$$

EXAMPLES OF INNER PRODUCT SPACES

1) Given $\lambda_1, \lambda_2, \dots, \lambda_n$ POSITIVE numbers, one can define an inner product in $\mathbb{R}^n$ like this:

$$\langle \vec{x}, \vec{y} \rangle := \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \dots + \lambda_n x_n y_n$$

2) Let $\mathbb{P}_n$ be the vector space of polynomials of degree at most n. Let $t_0, t_1, \dots, t_{n-1}$ be distinct numbers. Define

$$\langle p, q \rangle := p(t_0) q(t_0) + p(t_1) q(t_1) + \dots + p(t_{n-1}) q(t_{n-1}).$$

3) Let $\mathcal{C}[a,b]$ be the vector space of continuous functions on $[a,b]$. Define $\langle f, g \rangle := \int_a^b f(t) g(t)\, dt$.

4) Let $\mathcal{C}[a,b]$ be the vector space of continuous functions on $[a,b]$. Let $w: [a,b] \to \mathbb{R}$ be a POSITIVE function. Define $\langle f, g \rangle := \int_a^b f(t) g(t) w(t)\, dt$.

THEOREM: 1) The concepts of orthogonality, orthogonal complement, and orthogonal projection make sense when $\mathbb{R}^n$ is replaced by a vector space and the dot product by an inner product.

2) The Orthogonal Decomposition Theorem,
the Best Approximation Theorem and
the Pythagorean Theorem
are true, and the Gram-Schmidt algorithm works when $\mathbb{R}^n$ is replaced by a vector space and the dot product by an inner product.

THE MOST IMPORTANT INEQUALITIES IN MATHEMATICS

① THE CAUCHY-SCHWARZ INEQUALITY

$$|\langle \vec{v}, \vec{w} \rangle|^2 \leqslant \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle$$

We know $\langle \vec{u}, \vec{u} \rangle \geqslant 0$ and is 0 only if $\vec{u} = \vec{0}$.

$\langle \vec{v} - \vec{w}, \vec{v} - \vec{w} \rangle \geqslant 0$ $\qquad$ $\langle \vec{v} + \vec{w}, \vec{v} + \vec{w} \rangle \geqslant 0$

$\Rightarrow \langle \vec{v}, \vec{v} \rangle + \langle \vec{w}, \vec{w} \rangle \geqslant 2\langle \vec{v}, \vec{w} \rangle$ $\qquad$ $\Rightarrow \langle \vec{v}, \vec{v} \rangle + \langle \vec{w}, \vec{w} \rangle \geqslant -2\langle \vec{v}, \vec{w} \rangle$

$\Rightarrow |\langle \vec{v}, \vec{w} \rangle| \leqslant \dfrac{\langle \vec{v}, \vec{v} \rangle + \langle \vec{w}, \vec{w} \rangle}{2}$ for ANY CHOICE of $\vec{v}, \vec{w}$

$$\Rightarrow \underbrace{|\langle \lambda \vec{v}, \tfrac{1}{\lambda} \vec{w} \rangle|}_{\shortparallel \atop |\langle \vec{v}, \vec{w} \rangle|} \leqslant \frac{\langle \lambda \vec{v}, \lambda \vec{v} \rangle}{2} + \frac{1}{2} \langle \tfrac{1}{\lambda} \vec{w}, \tfrac{1}{\lambda} \vec{w} \rangle = f(\lambda)$$

What λ makes $f(\lambda)$ as small as possible?

$$|\langle \vec{v}, \vec{w} \rangle|^2 \leq \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle$$

$$|\langle \vec{v}, \vec{w} \rangle| = \frac{\langle \lambda \vec{v}, \lambda \vec{v} \rangle}{2} + \frac{1}{2} \left\langle \frac{1}{\lambda} \vec{w}, \frac{1}{\lambda} \vec{w} \right\rangle = f(\lambda) \quad ⊛$$

What λ makes $f(\lambda)$ as small as possible?

Find λ such that $f'(\lambda) = 0$.

$$f'(\lambda) = \lambda \langle \vec{v}, \vec{v} \rangle - \lambda^{-3} \langle \vec{w}, \vec{w} \rangle = 0 \iff \lambda = \frac{\sqrt{\langle \vec{w}, \vec{w} \rangle}}{\sqrt{\langle \vec{v}, \vec{v} \rangle}}$$

Plug this minimizing λ into ⊛

$$|\langle \vec{v}, \vec{w} \rangle| \leq \frac{1}{2} \sqrt{\langle \vec{v}, \vec{v} \rangle} \sqrt{\langle \vec{w}, \vec{w} \rangle} + \frac{1}{2} \sqrt{\langle \vec{v}, \vec{v} \rangle} \sqrt{\langle \vec{w}, \vec{w} \rangle}.$$

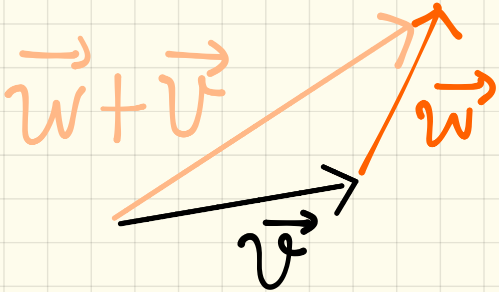
THE MOST IMPORTANT INEQUALITIES IN MATHEMATICS

① THE CAUCHY-SCHWARZ INEQUALITY

$$|\langle \vec{v}, \vec{w} \rangle|^2 \leqslant \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle$$

② THE TRIANGLE INEQUALITY

$$\|\vec{v}+\vec{w}\| \leqslant \|\vec{v}\| + \|\vec{w}\|$$



$$\|\vec{v}+\vec{w}\|^2 = \langle \vec{v}+\vec{w}, \vec{v}+\vec{w} \rangle$$

$$= \langle \vec{v}, \vec{v} \rangle + \langle \vec{w}, \vec{w} \rangle + 2\langle \vec{v}, \vec{w} \rangle$$

$$\overset{①}{\leqslant} \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\| \cdot \|\vec{w}\| = (\|\vec{v}\| + \|\vec{w}\|)^2$$

EXAMPLE: Consider the vector space

$$\mathcal{C}[0,2\pi] = \{f:[0,2\pi]\to\mathbb{R} \text{ continuous}\}$$

with the inner product

$$\langle f,g\rangle = \int_0^{2\pi} f(t)g(t)\,dt.$$

Then $\langle \cos mt, \sin nt\rangle = 0$ for any choice of integers m and n.

We need to verify

$$\int_0^{2\pi} \cos mt \sin nt \, dt \stackrel{?}{=} 0$$

$$e^{imt}e^{int} = (\cos mt \cos nt - \sin mt \sin nt) + i(\sin mt \cos nt + \sin nt \cos mt)$$

$$e^{-imt}e^{int} = (\cos mt \cos nt + \sin mt \sin nt) + i(-\sin mt \cos nt + \sin nt \cos mt)$$

Look only at the imaginary part.

EXAMPLE: Consider the vector space

$$\mathcal{C}[0,2\pi] = \{f:[0,2\pi] \to \mathbb{R} \text{ continuous}\}$$

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We need to verify

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$$e^{-imt}e^{int} = (\cos mt \cos nt + \sin mt \sin nt) + i(-\sin mt \cos nt + \sin nt \cos mt)$$

$$\operatorname{Im}\int_0^{2\pi} e^{i(m+n)t} + e^{i(-m+n)t}\,dt = 2\int_0^{2\pi} \cos mt \sin nt \; dt$$

$$\parallel$$

$$\operatorname{Im}\left\{ \frac{e^{i(m+n)t}}{m+n} + \frac{e^{i(-m+n)}}{-m+n} \Big|_0^{t=2\pi} \right\} = 0$$

PROBLEM: Consider the vector space

$$\mathcal{C}[-1,1] = \{f : [-1,1] \to \mathbb{R} \text{ continuous}\}$$

with the inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)\,dt.$$

Find an orthogonal basis for $\operatorname{span}\{1, t, t^2\}$.

We can use the Gram-Schmidt process.

We are given $\vec{u}_1 = 1$, $\vec{u}_2 = t$, $\vec{u}_3 = t^2$ and we look for $\vec{q}_1, \vec{q}_2, \vec{q}_3$ orthogonal with $\operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \operatorname{span}\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$.

$$\vec{q}_1 = \vec{u}_1$$

$$\vec{q}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{q}_1 \rangle}{\langle \vec{q}_1, \vec{q}_1 \rangle} \vec{q}_1$$

$$\vec{q}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{q}_1 \rangle}{\langle \vec{q}_1, \vec{q}_1 \rangle} \vec{q}_1 - \frac{\langle \vec{u}_3, \vec{q}_2 \rangle}{\langle \vec{q}_2, \vec{q}_2 \rangle} \vec{q}_2$$

PROBLEM: Consider the vector space

$$C[-1,1] = \{f : [-1,1] \to \mathbb{R} \text{ continuous}\}$$

with the inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)\,dt.$$

Find an orthogonal basis for span $\{1, t, t^2\}$.

$$\vec{u}_1 = 1, \quad \vec{u}_2 = t, \quad \vec{u}_3 = t^2$$

$$\vec{q}_1 = \vec{u}_1 = 1$$

$$\vec{q}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{q}_1 \rangle}{\langle \vec{q}_1, \vec{q}_1 \rangle}\vec{q}_1 = t - \frac{0}{2} \cdot 1 = t$$

$$\vec{q}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{q}_1 \rangle}{\langle \vec{q}_1, \vec{q}_1 \rangle}\vec{q}_1 - \frac{\langle \vec{u}_3, \vec{q}_2 \rangle}{\langle \vec{q}_2, \vec{q}_2 \rangle}\vec{q}_2 = t^2 - \frac{\frac{2}{3}}{2} 1 - \frac{0}{\frac{2}{3}} t$$

$$= t^2 - \frac{1}{3}$$

PROBLEM: Consider the vector space

$$\mathcal{C}[-1,1] = \{f : [-1,1] \to \mathbb{R} \text{ continuous}\}$$

with the inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)\,dt.$$

Find an orthogonal basis for span $\{1, t, t^2\}$.

Gram-Schmidt yielded $\{1, t, t^2 - \frac{1}{3}\}$.

CHECK ORTHOGONALITY:

$$\langle 1, t^2 - \tfrac{1}{3} \rangle \overset{?}{=} 0$$

$$\int_{-1}^{1} 1 \cdot (t^2 - \tfrac{1}{3})\,dt \overset{?}{=} 0$$

$$\Rightarrow \frac{t^3}{3} - \frac{t}{3} \Big|_{t=-1}^{t=1} = 0$$

$$\langle t, t^2 - \tfrac{1}{3} \rangle \overset{?}{=} 0$$

$$\int_{-1}^{1} t(t^2 - \tfrac{1}{3})\,dt \overset{?}{=} 0$$

$$\| $$

$$\int_{-1}^{1} t^3 - \frac{t}{3}\,dt = \frac{t^4}{4} - \frac{t^2}{6} \Big|_{-1}^{t=1} = 0$$

BONUS PROBLEM: Consider the vector space

$$V = \{ f: \mathbb{R} \to \mathbb{R} \text{ such that } \int_{-\infty}^{\infty} f(x)^2 e^{-x^2/2}\, dx < \infty \}$$

with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) e^{-x^2/2}\, dx .$$

A) Let $P_n(x) = e^{x^2/2} \left(\frac{d^n}{dx^n} e^{-x^2/2} \right)$

(that is, differentiate $e^{-x^2/2}$ n times and multiply the result by $e^{x^2/2}$). Show that $P_n(x)$ is a polynomial.

B) Using integration by parts, show that $P_0(x), P_1(x), \ldots$ are orthogonal.

C) Show that $\|P_n\| = n!$.

LESSON 36: DIAGONALIZATION OF SYMMETRIC MATRICES

THEOREM: Let A be a SYMMETRIC square matrix. Then

$$A = QDQ^T$$

for some DIAGONAL D and ORTHOGONAL Q.

REMARK: * "A is symmetric" means $A^T = A$.

** an orthogonal matrix is one whose columns are orthonormal.

THEOREM: Let A be a SYMMETRIC square matrix. Then

$$A = QDQ^T$$

for some DIAGONAL D and ORTHOGONAL Q.

1) The eigenvalues of A are real numbers.

For if $Av = zv$ and $\bar{v}^T v = 1$ and A is symmetric then $\bar{z} = z$, because

$$z = \bar{v}^T (zv) = \bar{v}^T A v$$

$$\bar{z} = v^T (\bar{z}\bar{v}) = v^T A \bar{v} \quad \longleftarrow \bar{A}\bar{v} = \bar{z}\bar{v} \text{ and } \bar{A} = A$$

$$= v^T A^T \bar{v}$$

$$= (\bar{v}^T A v)^T$$

$$= z$$

THEOREM: Let A be a SYMMETRIC square matrix. Then

$$A = QDQ^T$$

for some DIAGONAL D and ORTHOGONAL Q.

1) The eigenvalues of A are real numbers.

2) Eigenvectors associated to different eigenvalues are orthogonal.

For if $Av = \lambda v$, $Aw = \mu w$ and $\lambda \neq \mu$ then

$$v \cdot w \underset{\substack{\uparrow \\ \text{if } \lambda \neq 0}}{=} \lambda^{-1}(\lambda v)\cdot w = \lambda^{-1}(Av)\cdot w = \lambda^{-1} v \cdot Aw = \frac{\mu}{\lambda} v \cdot w$$

$$\Rightarrow \lambda\, v \cdot w = \mu\, v \cdot w \Rightarrow v \cdot w = 0$$

THEOREM: Let A be a SYMMETRIC square matrix. Then

$$A = QDQ^T$$

for some DIAGONAL D and ORTHOGONAL Q.

1) The eigenvalues of A are real numbers.

2) Eigenvectors associated to different eigenvalues are orthogonal.

3) If A is $n \times n$ then there are n LI eigenvectors.

THEOREM: Let A be a SYMMETRIC square matrix. Then

i) $$A = QDQ^T$$

for some DIAGONAL D and ORTHOGONAL Q.

ii) $A = \lambda_1 \vec{q_1}\vec{q_1}^T + \lambda_2 q_2 \vec{q_2}^T + \cdots + \lambda_n \vec{q_n}\vec{q_n}^T$, (spectral decomposition)

where $Q = [\vec{q_1} \cdots \vec{q_n}]$ and $A\vec{q_j} = \lambda_j \vec{q_j}$.

$$A = \begin{pmatrix} | & | & | \\ \vec{q_1} & \vec{q_2} & \vec{q_3} \\ | & | & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} - & \vec{q_1} & - \\ - & \vec{q_2} & - \\ - & \vec{q_3} & - \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | \\ \lambda_1\vec{q_1} & \lambda_2\vec{q_2} & \lambda_3\vec{q_3} \\ | & | & | \end{pmatrix} \begin{pmatrix} - & \vec{q_1} & - \\ - & \vec{q_2} & - \\ - & \vec{q_3} & - \end{pmatrix}$$

$$= +\lambda_1 \begin{pmatrix} | \\ \vec{q_1} \\ | \end{pmatrix} \begin{pmatrix} - & \vec{q_1} & - \end{pmatrix} + \lambda_2 \begin{pmatrix} | \\ \vec{q_2} \\ | \end{pmatrix} \begin{pmatrix} - & \vec{q_2} & - \end{pmatrix} + \lambda_3 \begin{pmatrix} | \\ \vec{q_3} \\ | \end{pmatrix} \begin{pmatrix} - & \vec{q_3} & - \end{pmatrix}$$

THEOREM: Let A be a SYMMETRIC square matrix. Then

i) $$A = QDQ^T$$

for some DIAGONAL D and ORTHOGONAL Q.

ii) $A = \lambda_1 \vec{q_1}\vec{q_1}^T + \lambda_2 q_2 \vec{q_2}^T + \cdots + \lambda_n \vec{q_n}\vec{q_n}^T$,

where $Q = [\vec{q_1} \cdots \vec{q_n}]$ and $A\vec{q_j} = \lambda_j \vec{q_j}$.

REMARK: The orthogonal projection of $\vec{v}$ onto $\vec{q_j}$ is $\vec{q_j}\vec{q_j}^T\vec{v}$.

$$\text{proj}_{\vec{q_j}}\vec{v} = \vec{q_j}(\vec{q_j}\cdot\vec{v}) = \underbrace{\vec{q_j}}_{n\times 1}\underbrace{(\vec{q_j}^T v)}_{1\times 1} = (\vec{q_j}\vec{q_j}^T)\vec{v}$$

EXAMPLE: orthogonally diagonalize the matrix

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}$$

STEP 1: find the eigenvalues of A.

It is easier to find the eigenvalues of 3A.

$$0 = \det(3A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 2-\lambda \\ -1 & -1 \end{vmatrix}$$

$$= (2-\lambda)^3 - (2-\lambda) - (2-\lambda) - 1 - 1 - (2-\lambda) = -\lambda^3 + 6\lambda^2 - 9\lambda$$

$$= -\lambda(\lambda^2 - 6\lambda + 9) = -\lambda(\lambda-3)(\lambda-3)$$

EXAMPLE: orthogonally diagonalize the matrix

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

STEP 1: find the eigenvalues of A.

The eigenvalues of $3A$ are $0, 3$ and 3

$\Longrightarrow$ the eigenvalues of A are $0, 1$ and 1.

STEP 2: find an orthonormal basis of eigenvectors

*) $A\vec{x} = \vec{0}$. Can take $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

**) $(3A - 3I)\vec{x} = \vec{0}$ need two orthogonal eigenvectors

$(3A-3I)\vec{x}=\vec{0}$ need two orthogonal eigenvectors

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A) Pick two LI eigenvectors

B) Gram-Schmidt

$$\vec{v}:=\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{w}:=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\vec{u}:=\vec{w}-\frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}-\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}=\begin{pmatrix} 1 \\ -1/2 \\ -1/2 \end{pmatrix}$$

VERIFY: $A\vec{v}=\vec{v}$, $A\vec{u}=\vec{u}$ and $\vec{u}\cdot\vec{v}=0$.

EXAMPLE: orthogonally diagonalize the matrix

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

STEP 1: find the eigenvalues of A. 0, 1 and 1

STEP2: find an orthonormal basis of eigenvectors

we have $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \end{pmatrix}$ orthogonal but not orthonormal

NORMALIZE:

$$Q := \begin{pmatrix} 1/\sqrt{3} & 0 & \sqrt{2/3} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}$$

EXAMPLE: orthogonally diagonalize the matrix

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

STEP 1: find the eigenvalues of A.

STEP 2: find an orthonormal basis of eigenvectors

STEP 3: write $A = QDQ^T$

$$A = \begin{pmatrix} 1/\sqrt{3} & 0 & \sqrt{2/3} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}$$

STEP 4: check if $AQ = QD$

LECTURE 25: COMPLEX NUMBERS (APPENDIX B)

"Number of the form $a+bi$ where
a, b are real numbers and
$i^2 = -1$"

A square matrix

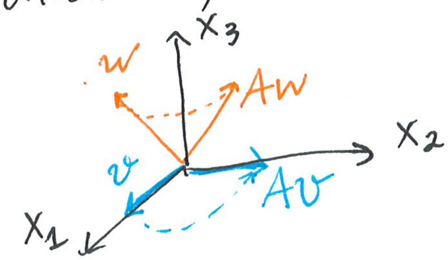
if λ is a real eigenvalue
there is a <u>1d subspace invariant by A</u>

if $a+bi$ is an eigenvalue (i.e. root of the charact. eq.) $b \neq 0$ then $a-bi$ is also an eigenvalue and there is a <u>2d</u> invariant subspace

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

eigenvalues
$1, i, -i$

↳ rotation by 90° in the x_1, x_2 plane



The rules of algebra:

$a + (b + c) = (a + b) + c$

$a \cdot (bc) = (a \cdot b) \cdot c$

$a + b = b + a$

$a \cdot b = b \cdot a$

$a + 0 = a$

$a \cdot 1 = a$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

① Algebraic view of complex numbers

"i is a symbol that 'solves' the equation $x^2 = -1$"

Ⓐ $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

$+$ $\quad\cdot$

$3 \cdot 5 := 3 + 3 + 3 + 3 + 3$
$= 5 + 5 + 5$

defined

Cannot solve $x + 2 = 0$.

Invent the symbol "-2" to 'solve' the equation.

Define $(-m)n = m(-n) = -mn$

and $(-m)(-n) = mn$

.) agrees with $\cdot$ on $\mathbb{N}$

$(-3)4 = -3 - 3 - 3 - 3$

..) the rules of algebra work

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

closed under $+$ and $\cdot$

rules of algebra work

can't solve $3x = 2$.

invent the symbol $\frac{m}{n}$ to 'solve' $n \cdot x = m$

<u>DEFINE</u> $+$ and $\cdot$ such that

A) they agree with $+$ and $\cdot$ in $\mathbb{Z}$

B) the rules of algebra work

$$\frac{m}{n} + \frac{p}{q} = \frac{mq + pn}{nq}$$

$$\frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$$

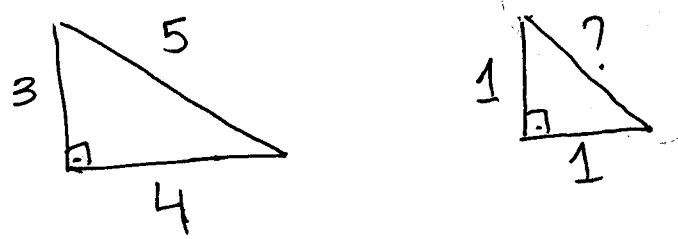
$\mathbb{Q} = \{$ all symbols $\pm \frac{m}{n}$

can't solve $x^2 = 2$

Just with fractions,

can't solve $x^2 = 2$

can't take limits



Can extend $\mathbb{Q}$ (fractions) to $\mathbb{R}$ (reals)

$\mathbb{R}$ = { all infinite digit. sequences

$a_0 . a_1 a_2 a_3 a_4 \ldots$

$\mathbb{Q}$ = finite decimal expansions

can define (tricky) $+$ and $\cdot$

A) agree with $+$ and $\cdot$ on $\mathbb{Q}$

B) the rules of algebra work

On $\mathbb{R}$, can't solve $x^2 = -1$.

Invent the symbol i to solve $x^2 = -1$

$\mathbb{C}$ = all symbols $a + bi$
$a, b \in \mathbb{R}$

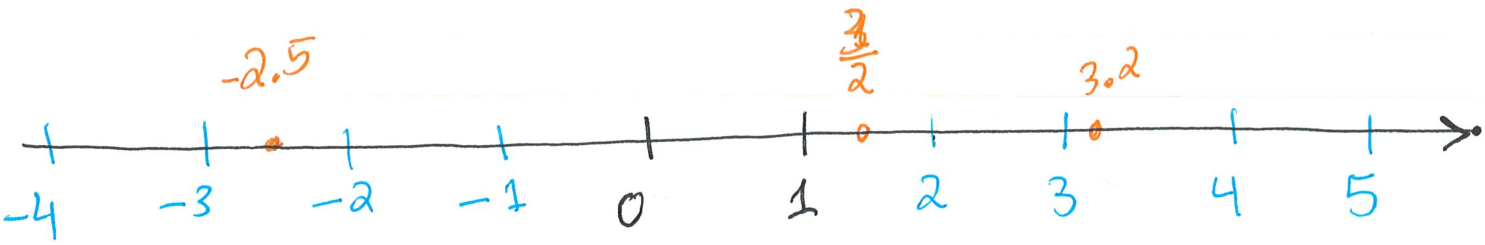
Define
$(a+bi) + (c+di) = (a+c) + (b+d)i$

$(a+bi)(c+di) = (ac-bd) + i(bc+ad)$

Then
A) agree with $+$ and $\cdot$ on $\mathbb{R}$

B) the rules of algebra work

② Geometric view of complex numbers



$\mathbb{Z}$

$\mathbb{Q}$

$\mathbb{R}$

line

ref. point 0

unit of measure 1

direction

How to see multiplication geometrically?

$M_2 : \mathbb{R} \longrightarrow \mathbb{R}$

"doubling"

$M_2 x \overset{\text{by def}}{=} 2x$

$M_{\frac{1}{2}}$ "halving"

$M_{-1} = ?$

reflecting

A) $M_a(x+y) = M_a x + M_a y$

$M_a(xy) = x M_a y$

M_a is <u>linear</u>

B) $M_a M_b = M_{ab}$

$M_{-1} M_{-1} = M_1$

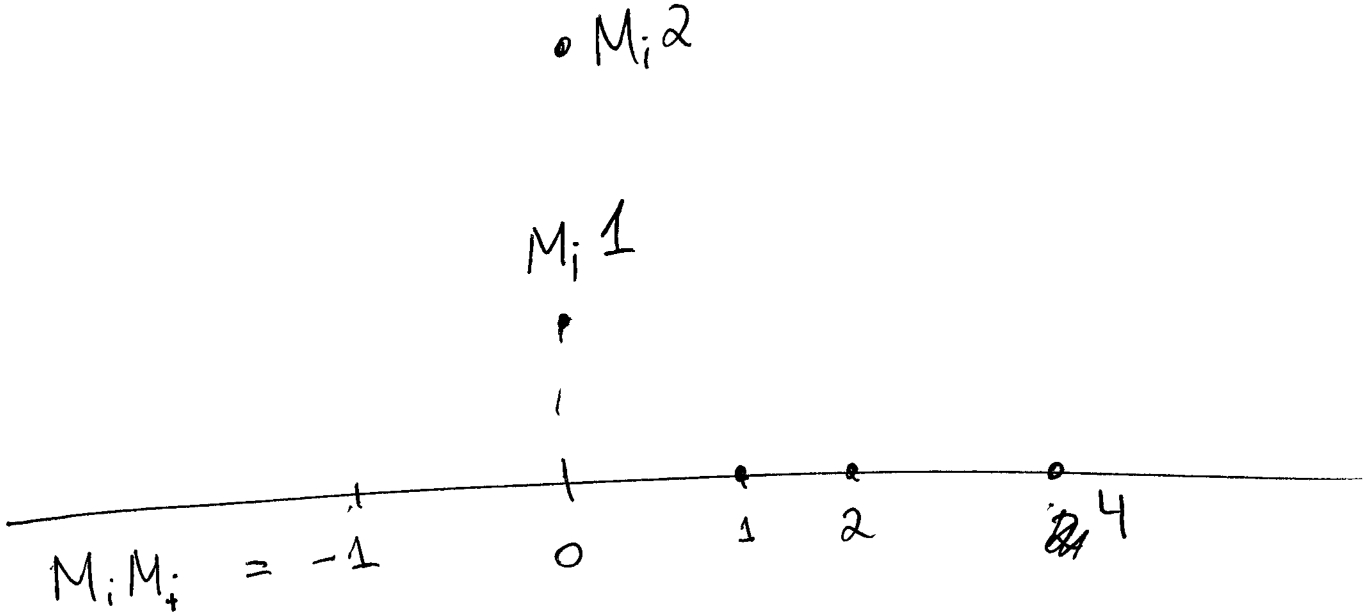
$x = M_{-1} M_{-1}$

How to solve $x^2 = -1$?

Reinterpret as $M_x M_x = M_{-1}$

Define M_i as rotation by 90° ↺

then $M_i M_i = M_{-1}$



Reinterpret M_a as a function on the plane and not on the line

$$M_{\cancel{a}2} \longleftrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$M_2 \longleftrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$M_i \longleftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$a+bi \longleftrightarrow \begin{bmatrix} a \\ b \end{bmatrix}$$

What should M_{a+bi} correspond to?

A) multiplication agrees with mult. on $\mathbb{R}$

B) the rules of algebra should work

~~eg~~ $M_{a+bi} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$M_a \longleftrightarrow \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

$$M_i \longleftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

~~(a+bi)~~

$$M_{a+bi}\begin{bmatrix} x \\ y \end{bmatrix} = M_a \begin{bmatrix} x \\ y \end{bmatrix} + M_{bi}\begin{bmatrix} x \\ y \end{bmatrix}$$

What is M_{bi} ?

~~Should have M_{bi}~~.

$$M_{bi} \longleftrightarrow \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

rotate by 90° and

~~multiply~~ dilate by b.

$$M_{a+bi} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}}_{M_a} + \underbrace{\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}}_{M_{bi}}$$

Define

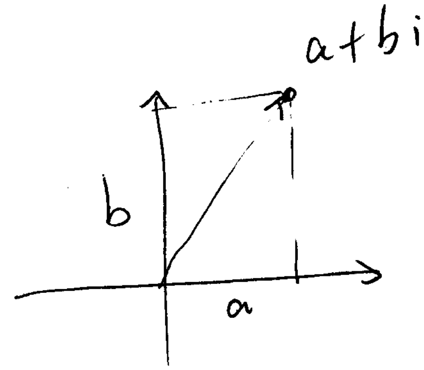
$$M_{a+ib} := \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

then $M_z M_w = M_{zw}$

A complex number can be

.) a symbol $a+bi$

..) a point in $\mathbb{R}^2$



∴) a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$