SECTION A1 | A Brief Review of Algebra

There are many techniques from elementary algebra that are needed in calculus. This appendix contains a review of such topics, and we begin by examining numbering systems.

The Real Numbers

An **integer** is a "whole number," either positive or negative. For example, 1, 2, 875, -15, -83, and 0 are integers, while $\frac{2}{3}$, 8.71, and $\sqrt{2}$ are not.

A **rational number** is a number that can be expressed as the quotient $\frac{a}{b}$ of two integers, where $b \neq 0$. For example, $\frac{2}{3}$, $\frac{8}{5}$, and $\frac{-4}{7}$ are rational numbers, as are

$$-6\frac{1}{2} = \frac{-13}{2} \qquad \text{and} \qquad 0.25 = \frac{25}{100} = \frac{1}{4}$$

Every integer is a rational number since it can be expressed as itself divided by 1. When expressed in decimal form, rational numbers are either terminating or infinitely repeating decimals. For example,

$$\frac{5}{8} = 0.625 \qquad \frac{1}{3} = 0.33\ldots \qquad \text{and} \qquad \frac{13}{11} = 1.181818\ldots$$

A number that cannot be expressed as the quotient of two integers is called an **irrational number.** For example,

$$\sqrt{2} \approx 1.41421356 \qquad \text{and} \qquad \pi \approx 3.14159265$$

are irrational numbers.

The rational numbers and irrational numbers form the **real numbers** and can be visualized geometrically as points on a **number line** as illustrated in Figure A.1.

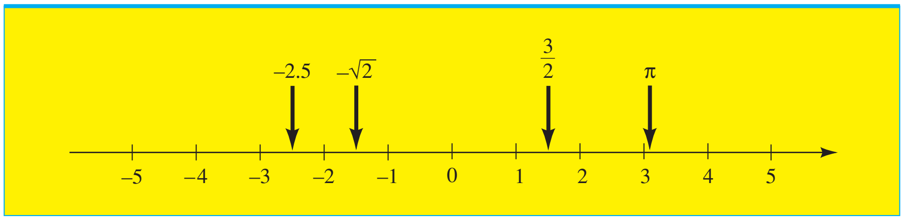


FIGURE A.1 The number line.

Inequalities

If a and b are real numbers and a is to the right of b on the number line, we say that ***a* is greater than *b*** and write $a > b$. If a is to the left of b, we say that ***a* is less than *b*** and write $a < b$ (Figure A.2). For example,

$$5 > 2 \qquad -12 < 0 \qquad \text{and} \qquad -8.2 < -2.4$$

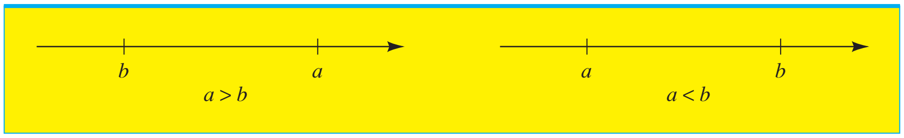


FIGURE A.2 Inequalities.

Moreover,

$$\frac{6}{7} < \frac{7}{8}$$

as you can see by noting that

$$\frac{6}{7} = \frac{48}{56} \qquad \text{and} \qquad \frac{7}{8} = \frac{49}{56}$$

The symbol $\geq$ stands for **greater than or equal to,** and the symbol $\leq$ stands for **less than or equal to.** Thus, for example,

$$-3 \geq -4 \qquad -3 \geq -3 \qquad -4 \leq -3 \qquad \text{and} \qquad -4 \leq -4$$

Intervals

A set of real numbers that can be represented on the number line by a line segment is called an **interval.** Inequalities can be used to describe intervals. For example, the interval $a \leq x < b$ consists of all real numbers x that are between a and b, including a but excluding b. This interval is shown in Figure A.3. The numbers a and b are known as the **endpoints** of the interval. The square bracket at a indicates that a is included in the interval, while the rounded bracket at b indicates that b is excluded.

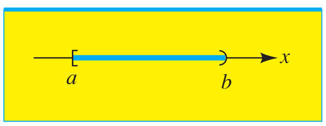


FIGURE A.3 The interval $a \leq x < b$.

Intervals may be finite or infinite in extent and may or may not contain either endpoint. The possibilities (including customary notation and terminology) are illustrated in Figure A.4.

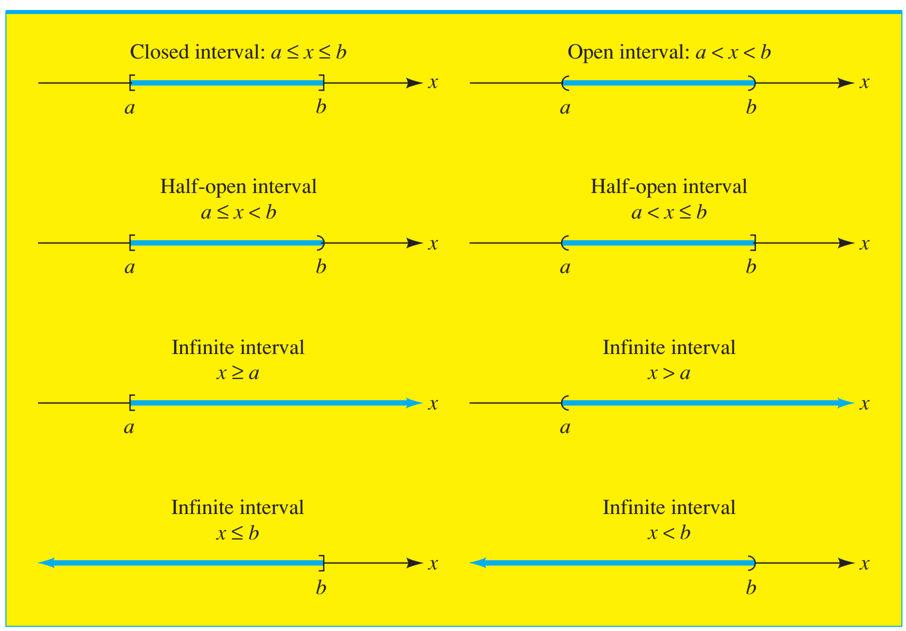
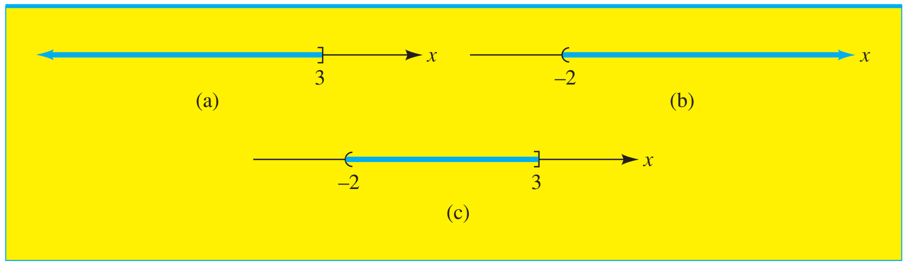


FIGURE A.4 Intervals of real numbers.

EXAMPLE A1.1

Use inequalities to describe these intervals.



Solution

a. $x \le 3$ **b.** $x > -2$ **c.** $-2 < x \le 3$

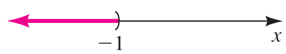
EXAMPLE A1.2

Represent each of these intervals as a line segment on a number line.

a. $x < -1$ **b.** $-1 \le x \le 2$ **c.** $x > 2$

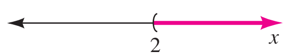
Solution

a.



b.

c.



Absolute Value

The **absolute value** of a real number x, denoted by $|x|$, is the distance from x to 0 on a number line. Since distance is always nonnegative, it follows that $|x| \ge 0$. For example,

$$|4| = 4 \qquad |-4| = 4 \qquad |0| = 0 \qquad |5 - 9| = 4 \qquad |\sqrt{3} - 3| = 3 - \sqrt{3}$$

Here is a general formula for absolute value.

Absolute Value ■ For any real number x, the absolute value of x is

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

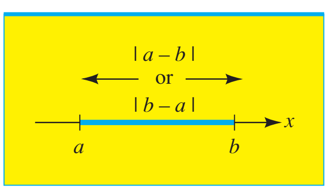


FIGURE A.5 The distance between a and $b = |a - b|$.

More generally, the distance between any two numbers a and b is the absolute value of their difference taken in either order, as illustrated in Figure A.5.

EXAMPLE A1.3

Find the distance on the number line between -2 and 3.

Solution

The distance between two numbers is the absolute value of their difference. Hence,

$$\text{Distance} = |-2 - 3| = |-5| = 5$$

The situation is illustrated in Figure A.6.

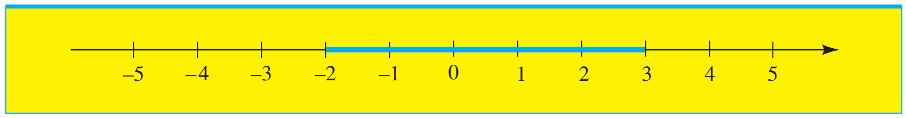


FIGURE A.6 Distance between -2 and 3.

Absolute Values and Intervals

The geometric interpretation of absolute value as distance can be used to simplify certain algebraic inequalities involving absolute values. Here is an example.

EXAMPLE A1.4

Find the interval consisting of all real numbers x such that $|x - 1| \le 3$.

Solution

In geometric terms, the numbers x for which $|x - 1| \le 3$ are those whose distance from 1 is less than or equal to 3. As illustrated in Figure A.7, these are the numbers that satisfy $-2 \le x \le 4$.

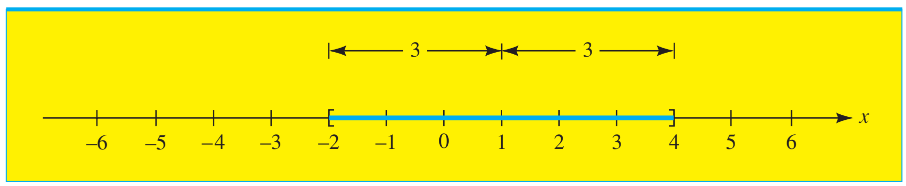


FIGURE A.7 The interval on which $|x - 1| \le 3$ is $-2 \le x \le 4$.

To find this interval algebraically, without relying on the geometry, rewrite the inequality $|x - 1| \le 3$ as

$$-3 \le x - 1 \le 3$$

and add 1 to each term to get

$$-3 + 1 \le x - 1 + 1 \le 3 + 1$$

or

$$-2 \le x \le 4$$

Exponential Notation

These rules define the expression a^x for $a > 0$ and all rational values of x.

Definition of a^x for $x \ge 0$ ■ Integer powers: If n is a positive integer, then

$$a^n = a \cdot a \cdots a$$

where the product $a \cdot a \cdots a$ contains n factors.
Fractional powers: If n and m are positive integers, then

$$a^{n/m} = (\sqrt[m]{a})^n = \sqrt[m]{a^n}$$

where $\sqrt[m]{}$ denotes the positive mth root.

Negative powers: $a^{-x} = \dfrac{1}{a^x}$

Zero power: $a^0 = 1$

EXAMPLE A1.5

Evaluate these expressions (without using your calculator).

a. $9^{1/2}$ **b.** $27^{2/3}$ **c.** $8^{-1/3}$ **d.** $\left(\dfrac{1}{100}\right)^{-3/2}$ **e.** 5^0

Solution

a. $9^{1/2} = \sqrt{9} = 3$

b. $27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9$

$= \sqrt[3]{(27)^2} = \sqrt[3]{729} = 9$

c. $8^{-1/3} = \dfrac{1}{8^{1/3}} = \dfrac{1}{\sqrt[3]{8}} = \dfrac{1}{2}$

d. $\left(\dfrac{1}{100}\right)^{-3/2} = 100^{3/2} = (\sqrt{100})^3 = 10^3 = 1{,}000$

e. $5^0 = 1$

Laws of Exponents

Exponents obey these useful laws.

Laws of Exponents ■ For a real number a, we have

The product law: $a^r a^s = a^{r+s}$

The quotient law: $\dfrac{a^r}{a^s} = a^{r-s}$ if $a \neq 0$

The power law: $(a^r)^s = a^{rs}$

The laws of exponents are illustrated in Examples A1.6 and A1.7.

EXAMPLE A1.6

Evaluate these expressions (without using a calculator).

a. $(2^{-2})^3$ **b.** $\dfrac{3^3}{3^{1/3}(3^{2/3})}$ **c.** $2^{7/4}(8^{-1/4})$

Solution

a. $(2^{-2})^3 = 2^{-6} = \dfrac{1}{2^6} = \dfrac{1}{64}$

b. $\dfrac{3^3}{3^{1/3}(3^{2/3})} = \dfrac{3^3}{3^{1/3+2/3}} = \dfrac{3^3}{3^1} = 3^2 = 9$

c. $2^{7/4}(8^{-1/4}) = 2^{7/4}(2^3)^{-1/4} = 2^{7/4}(2^{-3/4}) = 2^{7/4-3/4} = 2^1 = 2$

EXAMPLE A1.7

Solve each of these equations for n.

a. $\dfrac{a^5}{a^2} = a^n$ **b.** $(a^n)^5 = a^{20}$

Solution

a. Since $\dfrac{a^5}{a^2} = a^{5-2} = a^3$, it follows that $n = 3$.

b. Since $(a^n)^5 = a^{5n}$, it follows that $5n = 20$ or $n = 4$.

Factoring

To factor an expression is to write it as a product of two or more terms, called **factors.** Factoring is used to simplify complicated expressions and to solve equations and is based on the distributive law for addition and multiplication.

The Distributive Law ■ For any real numbers a, b, and c,

$$ab + ac = a(b + c)$$

The factoring techniques you will need in this book are illustrated in Examples A1.8 through A1.10.

Factoring Out Common Terms

EXAMPLE A1.8

Factor the expression $3x^4 - 6x^3$.

Solution

Since $3x^3$ is a factor of each of the terms in this expression, you can use the distributive law to "factor out" $3x^3$ and write

$$3x^4 - 6x^3 = 3x^3(x - 2)$$

EXAMPLE A1.9

Simplify the expression $10(1 - x)^4(x + 1)^4 + 8(x + 1)^5(1 - x)^3$.

Solution

The greatest common factor is $2(1 - x)^3(x + 1)^4$. Factor this out to get

$$10(1 - x)^4(x + 1)^4 + 8(x + 1)^5(1 - x)^3 = 2(1 - x)^3(x + 1)^4[5(1 - x) + 4(x + 1)]$$

Since no further factorization is possible, do the multiplication in the square brackets and combine the resulting terms to conclude that

$$\begin{aligned} 10(1 - x)^4(x + 1)^4 + 8(x + 1)^5(1 - x)^3 &= 2(1 - x)^3(x + 1)^4(5 - 5x + 4x + 4) \\ &= 2(1 - x)^3(x + 1)^4(9 - x) \end{aligned}$$

There are times in calculus when it is necessary to factor expressions involving radicals or fractional exponents. Here is an example of such a factorization.

EXAMPLE A1.10

Factor the expression

$$\frac{2}{3}x^{-1/3}(x + 1) + x^{2/3}$$

Solution

We find that

$$\begin{aligned}
\frac{2}{3}x^{-1/3}(x+1)+x^{2/3} &= \frac{\frac{2}{3}(x+1)}{x^{1/3}}+x^{2/3}\\
&= \frac{\frac{2}{3}(x+1)+x^{2/3}(x^{1/3})}{x^{1/3}} && \textit{put terms over the common denominator } x^{1/3} \textit{ and simplify}\\
&= \frac{\frac{2}{3}(x+1)+x}{x^{1/3}} = \frac{\frac{5}{3}x+\frac{2}{3}}{x^{1/3}} && \textit{since } x^{2/3}x^{1/3}=x\\
&= \frac{1}{3}(5x+2)x^{-1/3}
\end{aligned}$$

Simplifying Quotients by Factoring and Canceling

Example A1.11 illustrates how you can combine factoring and canceling to simplify certain types of quotients that arise frequently in calculus.

EXAMPLE | A1.11

Simplify the quotient

$$\frac{4(x+3)^4(x-2)^2-6(x+3)^3(x-2)^3}{(x+3)(x-2)^3}$$

Solution

First simplify the numerator to get

$$\begin{aligned}
\frac{4(x+3)^4(x-2)^2-6(x+3)^3(x-2)^3}{(x+3)(x-2)^3} &= \frac{2(x+3)^3(x-2)^2[2(x+3)-3(x-2)]}{(x+3)(x-2)^3}\\
&= \frac{2(x+3)^3(x-2)^2(2x+6-3x+6)}{(x+3)(x-2)^3}\\
&= \frac{2(x+3)^3(x-2)^2(12-x)}{(x+3)(x-2)^3}
\end{aligned}$$

and then "cancel" the common factor of $(x+3)(x-2)^2$ from the numerator and denominator to conclude that

$$\frac{4(x+3)^4(x-2)^2-6(x+3)^3(x-2)^3}{(x+3)(x-2)^3} = \frac{2(x+3)^2(12-x)}{x-2}$$

Summation Notation

Sums of the form

$$a_1+a_2+\cdots+a_n$$

appear in Chapters 5 and 6. To describe such a sum, it suffices to specify the general term a_j and to indicate that n terms of this form are to be added, starting with the term in which $j = 1$ and ending with the term in which $j = n$. It is customary to use the Greek letter Σ (sigma) to denote summation and to express the sum compactly as follows.

Summation Notation ■ The sum of the numbers $a_1 \cdots a_n$ is given by

$$a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j$$

The use of summation notation is illustrated in Examples A1.12 and A1.13.

EXAMPLE A1.12

Use summation notation to represent these sums.

a. $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64$

b. $(1 - x_1)^2 \Delta x + (1 - x_2)^2 \Delta x + \cdots + (1 - x_{15})^2 \Delta x$

Solution

a. This is a sum of 8 terms of the form j^2, starting with $j = 1$ and ending with $j = 8$. Hence,

$$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 = \sum_{j=1}^{8} j^2$$

b. The jth term of this sum is $(1 - x_j)^2 \Delta x$. Hence,

$$(1 - x_1)^2 \Delta x + (1 - x_2)^2 \Delta x + \cdots + (1 - x_{15})^2 \Delta x = \sum_{j=1}^{15} (1 - x_j)^2 \Delta x$$

EXAMPLE A1.13

Evaluate these sums.

a. $\displaystyle\sum_{j=1}^{4} (j^2 + 1)$

b. $\displaystyle\sum_{j=1}^{3} (-2)^j$

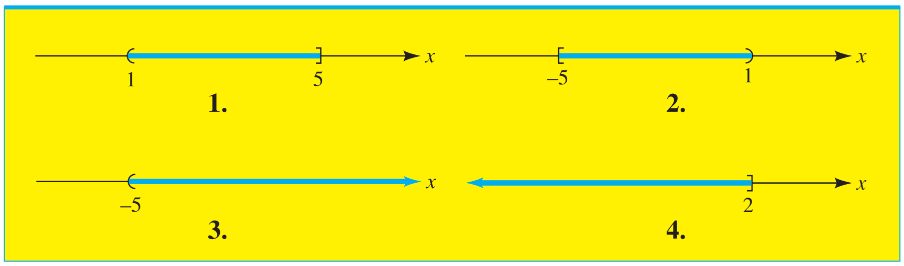
Solution

a.
$$\sum_{j=1}^{4} (j^2 + 1) = (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) = 2 + 5 + 10 + 17 = 34$$

b.
$$\sum_{j=1}^{3} (-2)^j = (-2)^1 + (-2)^2 + (-2)^3 = -2 + 4 - 8 = -6$$

PROBLEMS | A1

INTERVALS *In Problems 1 through 4, use inequalities to describe the indicated interval.*



In Problems 5 through 8, represent the given interval as a line segment on a number line.

5. $x \geq 2$

6. $-6 \leq x < 4$

7. $-2 < x \leq 0$

8. $x > 3$

DISTANCE *In Problems 9 through 12, find the distance on the number line between the given pair of real numbers.*

9. 0 and -4

10. 2 and 5

11. -2 and 3

12. -3 and -1

ABSOLUTE VALUE AND INTERVALS *In Problems 13 through 18, find the interval or intervals consisting of all real numbers x that satisfy the given inequality.*

13. $|z| \leq 3$

14. $|x - 2| \leq 5$

15. $|x + 4| \leq 2$

16. $|1 - x| < 3$

17. $|x + 2| \geq 5$

18. $|x - 1| > 3$

EXPONENTIAL NOTATION *In Problems 19 through 26, evaluate the given expression without using a calculator.*

19. 5^3

20. 2^{-3}

21. $16^{1/2}$

22. $36^{-1/2}$

23. $8^{2/3}$

24. $27^{-4/3}$

25. $\left(\frac{1}{4}\right)^{1/2}$

26. $\left(\frac{1}{4}\right)^{-3/2}$

In Problems 27 through 34, evaluate the given expression without using a calculator.

27. $\dfrac{2^5(2^2)}{2^8}$

28. $\dfrac{3^4(3^3)}{(3^2)^3}$

29. $\dfrac{2^{4/3}(2^{5/3})}{2^5}$

30. $\dfrac{5^{-3}(5^2)}{(5^{-2})^3}$

31. $\dfrac{2(16^{3/4})}{2^3}$

32. $\dfrac{\sqrt{27}\,(\sqrt{3})^3}{9}$

33. $[\sqrt{8}\,(2^{5/2})]^{-1/2}$

34. $[\sqrt{27}\,(3^{5/2})]^{1/2}$

In Problems 35 through 42, solve the given equation for n. (Assume $a > 0$ and $a \neq 1$.)

35. $a^3a^7 = a^n$

36. $\dfrac{a^5}{a^2} = a^n$

37. $a^4a^{-3} = a^n$

38. $a^2a^n = \dfrac{1}{a}$

39. $(a^3)^n = a^{12}$

40. $(a^n)^5 = \dfrac{1}{a^{10}}$

41. $a^{3/5}a^{-n} = \dfrac{1}{a^2}$

42. $(a^n)^3 = \dfrac{1}{\sqrt{a}}$

SIMPLIFYING EXPRESSIONS *In Problems 43 through 54, factor and simplify the given expression as much as possible.*

43. $x^5 - 4x^4$

44. $3x^3 - 12x^4$

45. $100 - 25(x - 3)$

46. $60 - 20(4 - x)$

47. $8(x + 1)^3(x - 2)^2 + 6(x + 1)^2(x - 2)^3$

48. $12(x + 3)^5(x - 1)^3 - 8(x + 3)^6(x - 1)^2$

49. $x^{-1/2}(2x + 1) + 4x^{1/2}$

50. $x^{-1/4}(3x + 5) + 4x^{3/4}$

51. $\dfrac{(x + 3)^3(x + 1) - (x + 3)^2(x + 1)^2}{(x + 3)(x + 1)}$

52. $\dfrac{3(x - 2)^2(x + 1)^2 - 2(x - 2)(x + 1)^3}{(x - 2)^4}$

53. $\dfrac{4(1 - x)^2(x + 3)^3 + 2(1 - x)(x + 3)^4}{(1 - x)^4}$

54. $\dfrac{6(x + 2)^5(1 - x)^4 - 4(x + 2)^6(1 - x)^3}{(x + 2)^8(1 - x)^2}$

SUMMATION NOTATION *In Problems 55 through 58, evaluate the given sum.*

55. $\displaystyle\sum_{j=1}^{4}(3j + 1)$

56. $\displaystyle\sum_{j=1}^{5}j^2$

57. $\displaystyle\sum_{j=1}^{10}(-1)^j$

58. $\displaystyle\sum_{j=1}^{5}2^j$

In Problems 59 through 64, use summation notation to represent the given sum.

59. $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6}$

60. $3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30$

61. $2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + 2x_6$

62. $1 - 1 + 1 - 1 + 1 - 1$

63. $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8$

64. $x - x^2 + x^3 - x^4 + x^5$

65. ECOLOGY The atmosphere above each square centimeter of the earth's surface weighs 1 kilogram (kg).

a. Assuming the earth is a sphere of radius $R = 6{,}440$ km, use the formula $S = 4\pi R^2$ to calculate the surface area of the earth and then find the total mass of the atmosphere.

b. Oxygen occupies approximately 22% of the total mass of the atmosphere, and it is estimated that plant life produces approximately 0.9×10^{13} kg of oxygen per year. If none of this oxygen were used up by plants or animals (or combustion), how long would it take to build up the total mass of oxygen in the atmosphere (part a)?*

*Adapted from a problem in E. Batschelet, *Introduction to Mathematics for Life Scientists*, 2nd ed., New York: Springer-Verlag, 1979, p. 31.