MA 108

Math—The Language of Engineers

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School of Electrical and Computer Engineering

October 14, 2010

Outline

- What is math?

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- Some pure math problems

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- Engineering challenge problems

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- Some cool math applications

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- Conclusions

What Is Math?

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- "Mathematical discoveries have come both from the attempt to describe the natural world and from the desire to arrive at a form of inescapable truth from careful reasoning"—Kenyon College Math Department Web Page

What Is Math Good For?

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- engineering

Pure Math vs. Applied Math

- The Millennium Prize Problems

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- http://www.claymath.org/millennium/

Engineering Grand Challenges

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- http://www.engineeringchallenges.org/cms/8996/9221.aspx

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- What does "best" mean?

Objective function

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- $\Omega =$ set of all possible choices. (*Feasible set*)

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- f is a real-valued function on n variables,

$$f : \mathbb{R}^n \to \mathbb{R}$$

Example

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$$f = f(x_1, x_2) = x_1 x_2 + 7,$$

an example of a real-valued function of two variables,

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Optimization problem

-
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- Analytically

- Numerically

Example: Linear regression

- Given points on the plane:

$$(t_0, y_0), \ldots, (t_n, y_n)$$

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- Best = minimize the average squared error.

Minimizing the average squared error
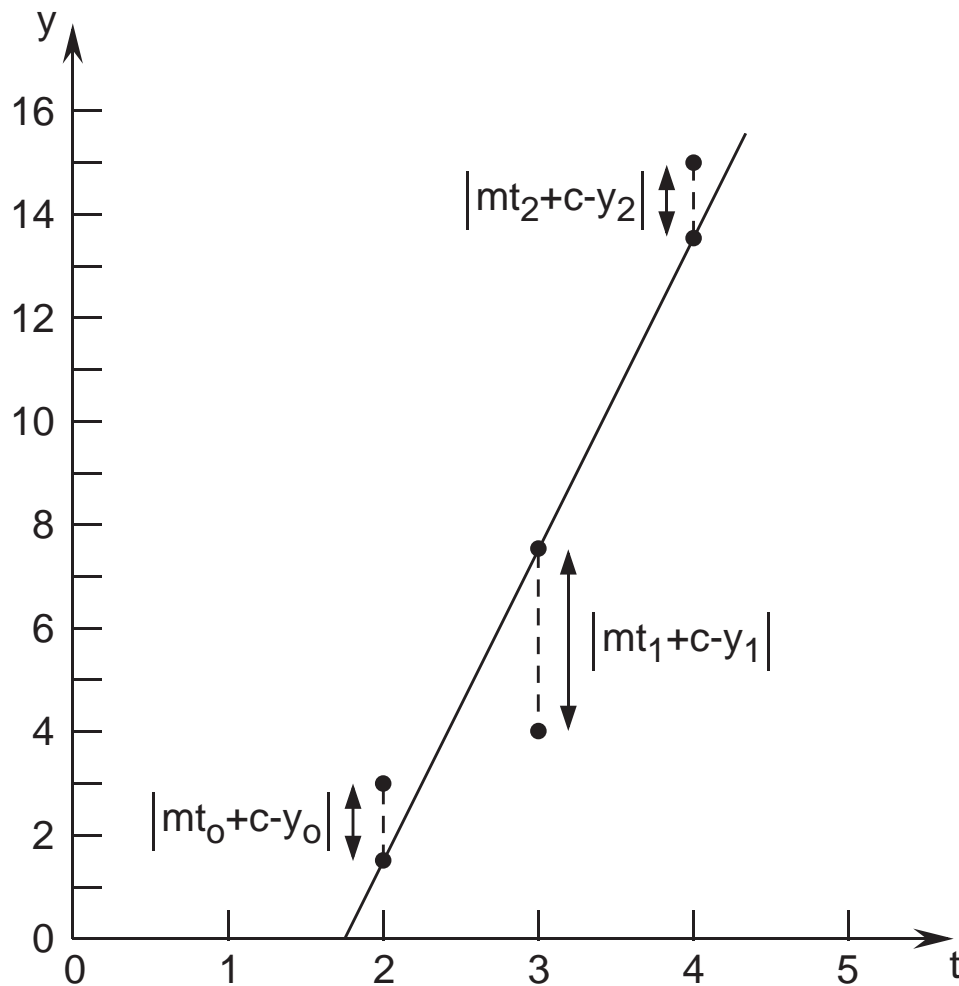
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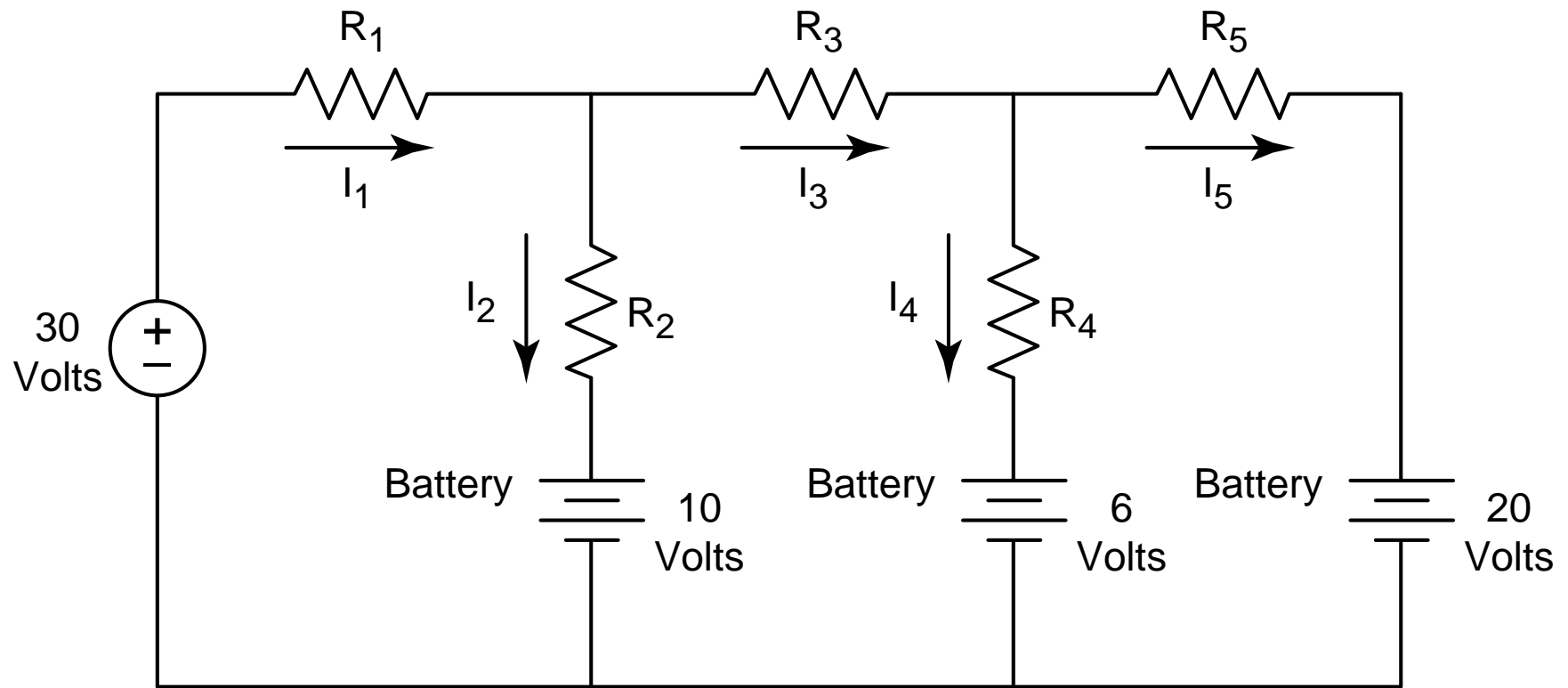- Solution: In this case we can find the solution analytically (using least-squares theory).

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- Related application: system identification.

Battery charger circuit

Charger circuit specifications

Current	I_1	I_2	I_3	I_4	I_5
Upper Limit (Amps)	4	3	3	2	2
Lower Limit (Amps)	0	0	0	0	0

Design objective

Find $I_1, \ldots, I_5$ to maximize power transferred to batteries, that is,

$$\max \quad 10I_2 + 6I_4 + 20I_5$$

$$\text{subject to} \quad I_1 = I_2 + I_3$$

$$I_3 = I_4 + I_5$$

$$I_1 \leq 4$$

$$I_2 \leq 3$$

$$I_3 \leq 3$$

$$I_4 \leq 2$$

$$I_5 \leq 2,$$

$$I_1, I_2, I_3, I_4, I_5 \geq 0.$$

Solving example problem

- This is a *linear programming problem.*

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- Solution: Can use the simplex algorithm—see MA 511.

Model-based Predictive Control

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- The idea behind this approach can be explained using an example of driving a car

# MPC—analogy with driving a car

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MPC—analogy with driving a car

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- Based on the prediction, the driver adjusts the driving direction.
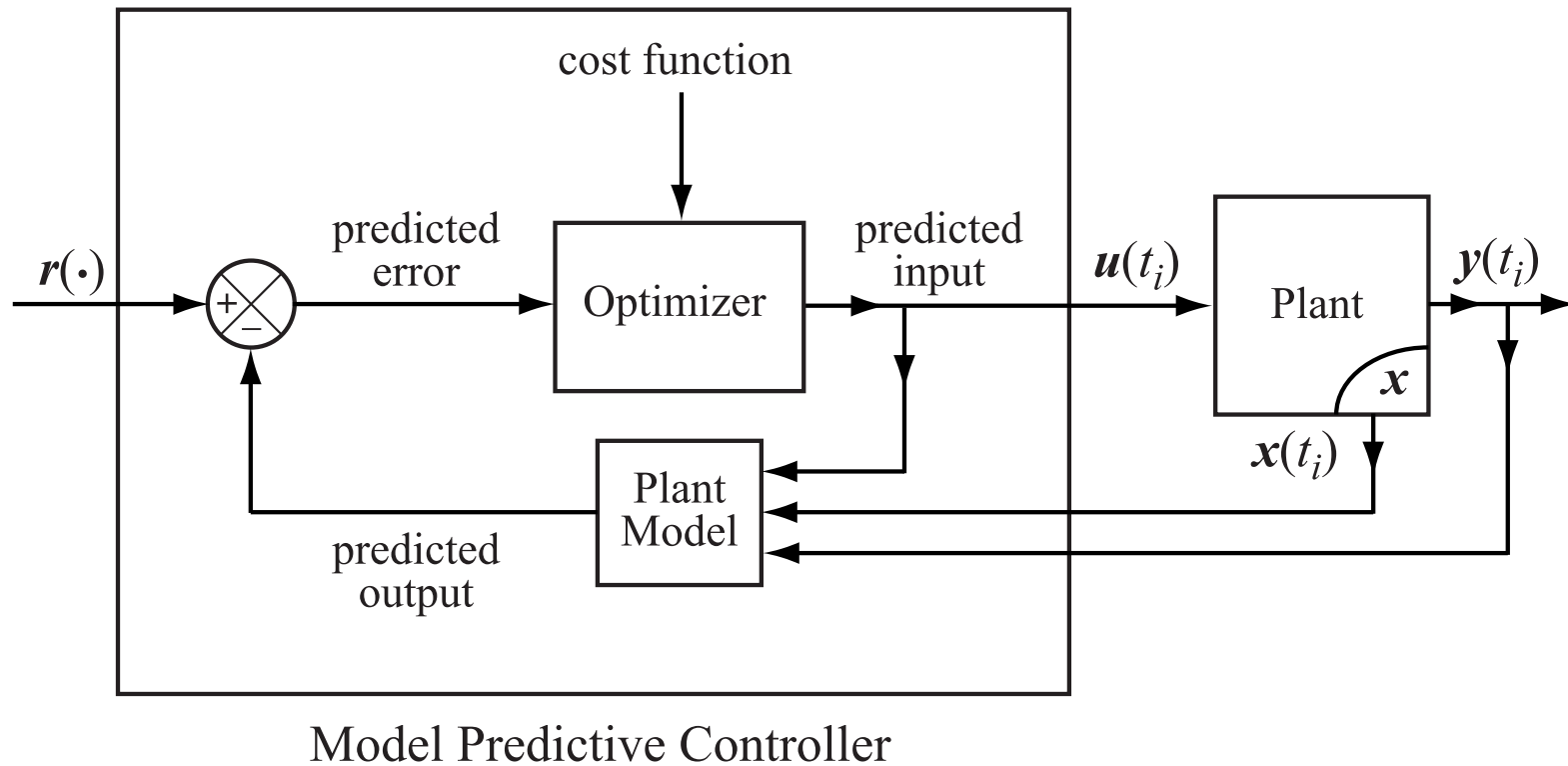
MPC illustration
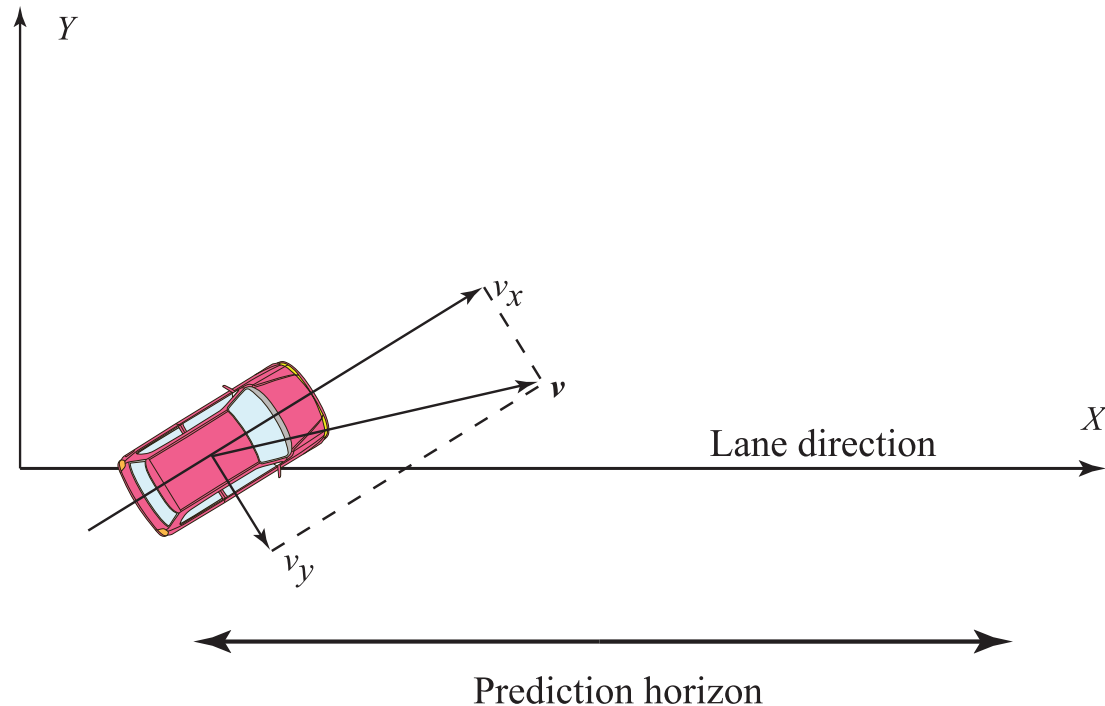
The driver predicts future travel direction based on the current state of the car and the current position of the steering wheel

MPC construction

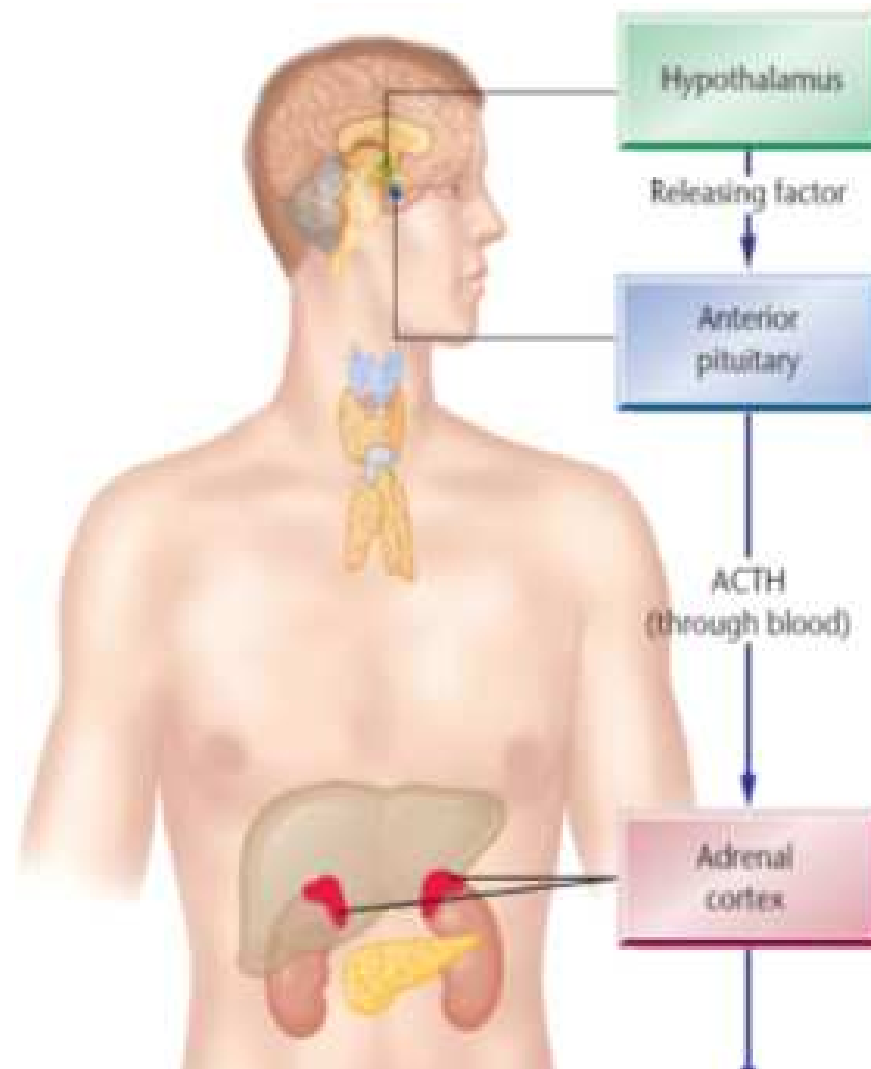Model Predictive Controller
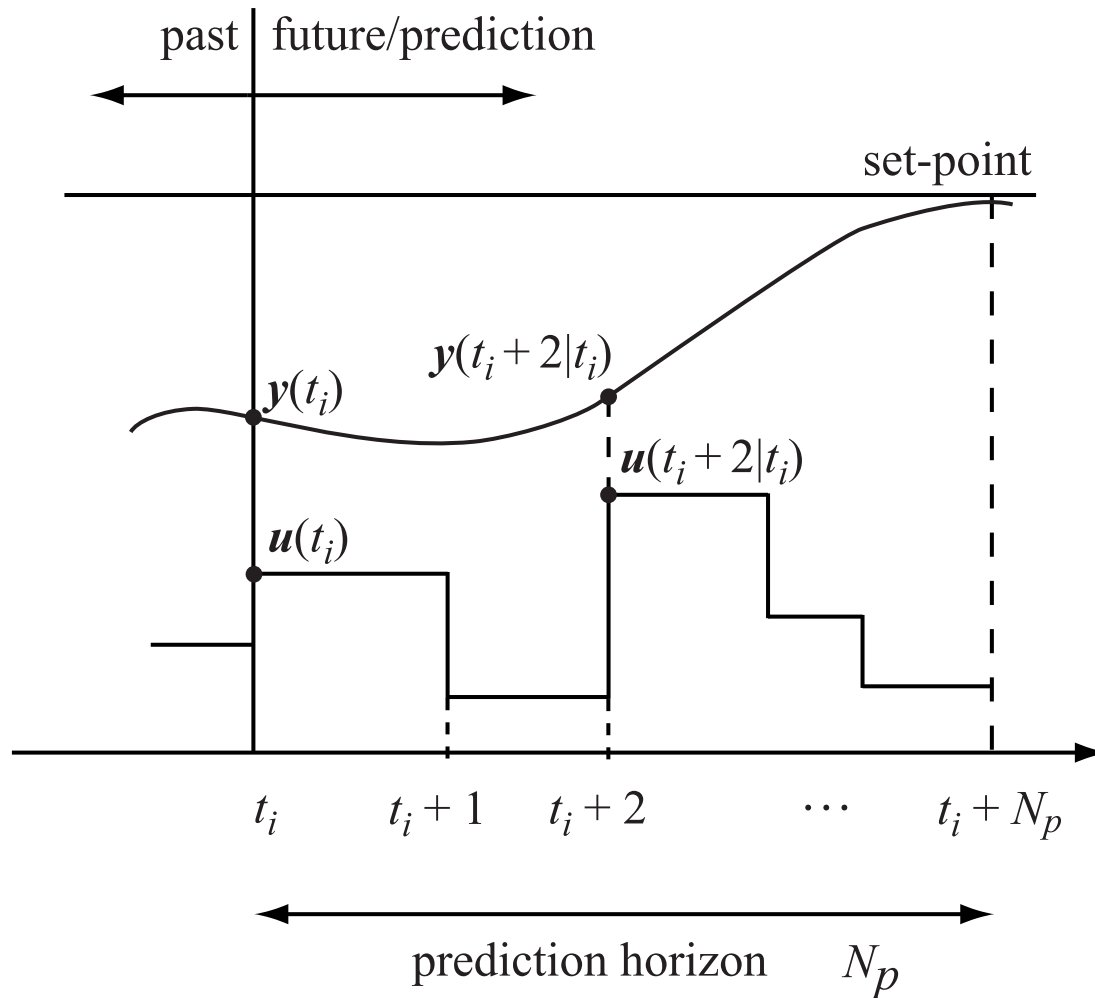
- The hypothalamic-pituitary-adrenal—HPA

Hypothalamic-pituitary-adrenal axis

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- The HPA axis is a set of interactions between the hypothalamus (a part of the brain), the pituitary gland (also part of the brain) and the adrenal or suprarenal glands (at the top of each kidney.)

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- It is also a major part of the system that controls our reaction to stress, trauma and injury.

HPA axis therapeutic correction

- A problem related to human health—how optimization can be used to find a therapeutic strategy

HPA axis therapeutic correction

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- http://www.ncbi.nlm.nih.gov/pmc/articles/PMC2613

Math and Soccer?

- http://www.bbc.co.uk/news/science-environment-11153466

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- http://iopscience.iop.org/1367-2630/12/9/093004/pdf/1367-$2630_1 2_{90} 93004.pdf$

Conclusions

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- Need to consider the moral consequences of what we do!