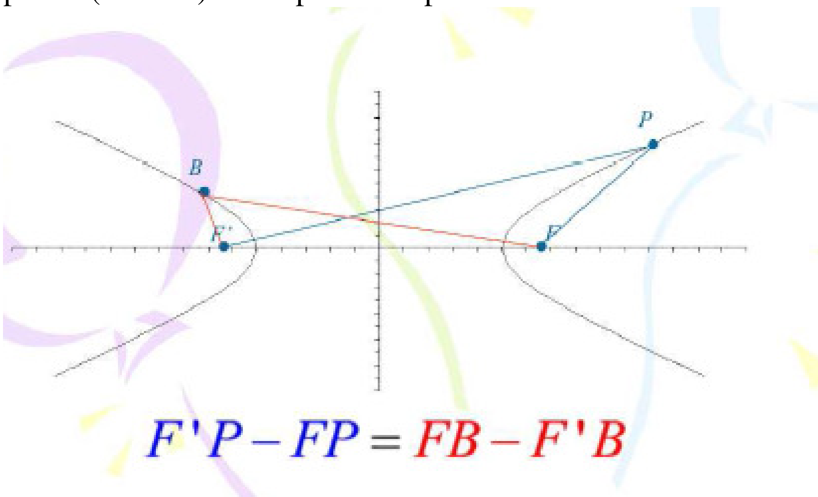
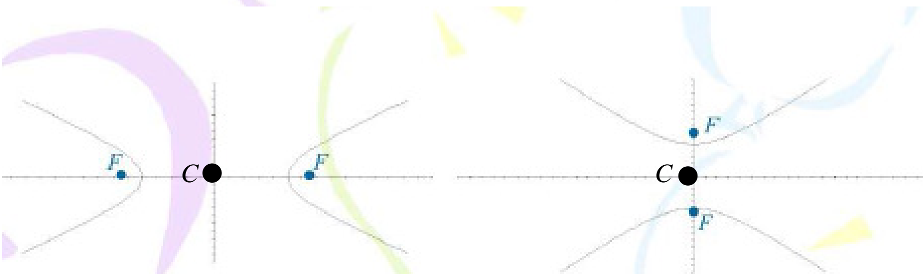
Hyperbolas

A **hyperbola** is the set of all points in a plane, the **difference** of whose distances from two fixed points (the **foci**) in the plane is a positive constant.



There are two 'branches' of a hyperbola. If the foci are on a horizontal line, the 'branches' are opening left and right. If the foci are on a vertical line, the 'branches' are opening up and down. The foci will lie 'inside' the 'branches'.

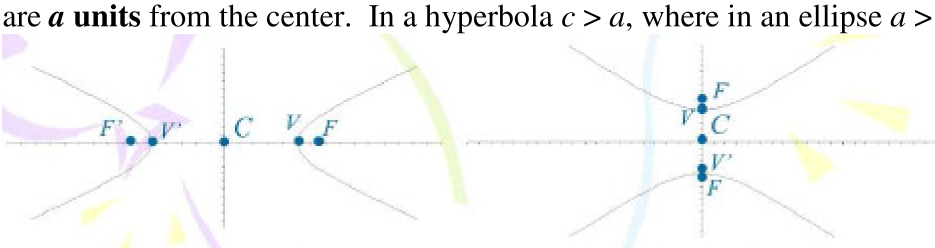


Hyperbola with a Horizontal Transverse Axis

Hyperbola with a Vertical Transverse Axis

The midpoint of the two foci is the **center** of the hyperbola. (Centers are marked with C.) The foci are ***c* units** from the center.

The points where the hyperbola intersects the line joining the foci are the **vertices**. The vertices are ***a* units** from the center. In a hyperbola $c > a$, where in an ellipse $a > c$.

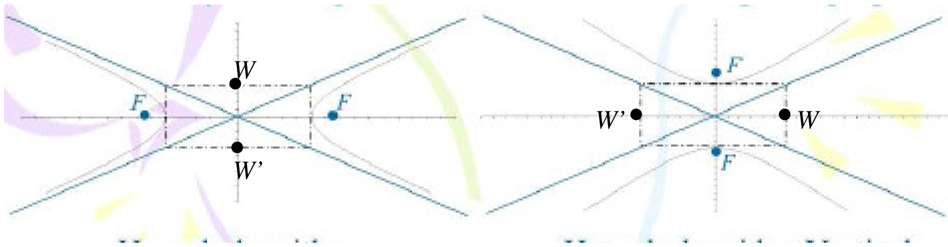


There are two axes:

1) The line segment V'V is the **transverse axis**. The foci lie beyond the endpoints of the transverse axis. The length of a transverse axis is $2a$.
2) The graph does not cross the other axis, the **conjugate axis**. Its endpoints, W and W', are not points on the hyperbola, however, are very important in creating an auxiliary rectangle that assists in sketching the graph. The length of a conjugate axis is $2b$.

(See the next page.)

Hyperbolas



Note: We cannot call one axis a major axis and the other a minor axis, because major implies larger and minor implies smaller. The length of the transverse axis may be larger or smaller than the length of the conjugate axis.

An ellipse equation has a + and $c^2 = a^2 - b^2$.

A hyperbola equation has a – and $c^2 = a^2 + b^2$.

Standard Equation of an Hyperbola with Center at the Origin:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad or \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Where $a > b$ or $b > a$ or $a = b$

The length of the transverse axis is $2a$

The length of the conjugate axis is $2b$

The foci are a distance c from the origin, where $c^2 = a^2 + b^2$

a^2 is always under the positive term.

a may be larger, equal, or may be smaller than *b*.

If the *x* term is positive, the branches are left and right; y term positive, then the branches are up and down.

The lines $y = \pm\frac{b}{a}x$ or $y = \pm\frac{a}{b}x$ are the **asymptotes** for the hyperbola. These asymptotes serve as excellent guides for sketching the graph. These asymptotes are the diagonals of the rectangle formed when using the vertices (endpoints of the transverse axis) and the endpoints of the conjugate axis.

For a hyperbola with a horizontal transverse axis: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **(Center at origin.)**

$y = \pm\frac{b}{a}x$ is the asymptotes for the hyperbola.

For a hyperbola with a vertical transverse axis: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ **(Center at origin.)**

$y = \pm\frac{a}{b}x$ is the asymptotes for the hyperbola.

Remember slope = $\frac{\triangle y}{\triangle x} = \frac{rise}{run}$

A convenient way to sketch the asymptotes is to draw an auxiliary rectangle using V'V and W'W as the corners of the rectangle. Extend the two diagonals of the rectangle to the edges of the graph. The hyperbola is then sketched, through the vertices, using the asymptotes as guides. The two parts that make up the hyperbola are called the right branch and the left branch of the hyperbola or the upper branch and lower branch.

Hyperbolas

Find the vertices, the foci, the endpoints of the conjugate axis, and the equations of the asymptotes of the hyperbola. Sketch its graph, showing the asymptotes and the foci.

A $\frac{y^2}{49} - \frac{x^2}{16} = 1$

B $\frac{x^2}{4} - \frac{y^2}{25} = 1$

C $4y^2 - x^2 = 4$

For a hyperbola with a horizontal transverse axis and center at (*h, k*):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$y - k = \pm\frac{b}{a}(x-h)$ for equations of asymptotes

$\rightarrow y = \pm\frac{b}{a}(x-h) + k$

For a hyperbola with a vertical transverse axis and center at (*h, k*):

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$y - k = \pm\frac{a}{b}(x-h)$ for equation of asymptotes

$\rightarrow y = \pm\frac{a}{b}(x-h) + k$

Hyperbolas

D $\frac{(x-2)^2}{16}-\frac{(y+3)^2}{25}=1$

E $4y^2-x^2+40y-4x+60=0$

Hyperbolas

Find an equation for the hyperbola.

F V(0, ±4), F(0, ±6)

G V(-2, -2) V'(-2, 4)
F(-2, 7) F'(-2, -5)

Find an equation for the hyperbola that has its **center at the origin** and satisfies the given conditions.

H Focus F(±5, 0)
Vertex V(±2, 0)

I Focus F(0, ±4)
Vertex V(0, ±3)

Hyperbolas

J Vertex V(0, ±6)
Passing through P(3, 9)

K Vertex V(±4, 0)
Asymptotes $y = \pm\frac{3}{2}x$

L y-intercepts ±7
Asymptotes $y = \pm 4x$

M Horizontal transverse axis of length 18
Conjugate axis of length 20

Hyperbolas

Equations of Conic Sections:

	Standard Form	General Form
Circle	$(x-h)^2+(y-k)^2=r^2$	$ax^2+ay^2+cx+dy+e=0 \quad a=a$
Example	$(x-2)^2+(y+8)^2=25$	$x^2+y^2-4x+16y+43=0$
Parabola	$(x-h)^2=4p(y-k)$ or $(y-k)^2=4p(x-h)$	$y=ax^2+bx+c$ or $x=ay^2+by+c$
Examples	$(y-2)^2=12(x+1)$ $(x+4)^2=\frac{1}{3}y$	$x=\frac{1}{12}y^2-\frac{1}{3}y-\frac{2}{3}$ $y=3x^2+24x+48$
Ellipse	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ or $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	$ax^2+by^2+cx+dy+e=0 \quad a\neq b, a>0, b>0$
Examples	$\frac{x^2}{4}+\frac{y^2}{9}=1$ $\frac{(x-2)^2}{16}+\frac{(y-1)^2}{4}=1$	$9x^2+4y^2=36$ $x^2+4y^2-4x-8y-8=0$
Hyperbolas	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$	$ax^2-by^2+cx+dy+e=0$ or $ay^2-bx^2+cy+dx+e=0 \quad a>0,\ b>0$
Examples	$\frac{x^2}{9}-\frac{(y+1)^2}{4}=1$ $\frac{(y-3)^2}{4}-\frac{(x+2)^2}{25}=1$	$4x^2-9y^2+18y-27=0$ $25y^2-4x^2-150y-16x+109=0$

Identify each equation below as a circle, parabola, ellipse, or hyperbola.

L $\frac{1}{2}(x-2)^2=y+6$

M $x^2-4y^2=16$

N $2x^2+8y^2-6x+16y+2=0$

O $x=2y^2-5$

P $x^2+y^2-4x-7=0$

Q $4y^2-x^2-8y-4x-4=0$

Hyperbolas