The Inverse Trigonometric Functions

Find θ in the interval [0, 2π) for each statement.

$\tan\theta = \frac{1}{\sqrt{3}}$ $\qquad$ $\sin\theta = \frac{\sqrt{3}}{2}$ $\qquad$ $\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{6}, \frac{7\pi}{6}$ $\qquad$ $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ $\qquad$ $\theta = \frac{\pi}{4}, \frac{7\pi}{4}$

While this is true for equations with the directions 'Find all solutions of the equation in the interval [0, 2π)', today we are going to define the **inverse trigonometric functions**.

INVERSE TRIGONOMETRIC FUNCTIONS:

Remember that a function must pass a vertical line test. And, for each *x*, there is only one *y*. For a trigonometric function, each angle yields only one trig value (a ratio). For an inverse trigonometric function, each trig value (ratio) would have to yield only one angle. However, we can see from above, a trig value, such as $\frac{1}{\sqrt{3}}$ can correspond to more than one angle, for example both $\frac{\pi}{6}$ and $\frac{7\pi}{6}$. Your calculator is 'set up' to return only one angle for each trig value. Remember that the inverse sine key or inverse tangent key only return a value in quadrants I or IV. The inverse cosine key only returns an angle in quadrants I or II.

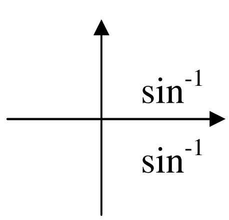
Notation for Inverse Trigonometric Functions:

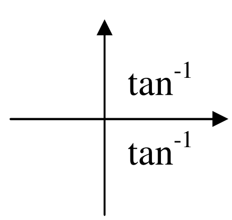
$y = \arcsin(x) = \sin^{-1}(x) = \theta$ $\qquad$ $y = \arccos(x) = \cos^{-1}(x) = \theta$ $\qquad$ $y = \arctan(x) = \tan^{-1}(x) = \theta$

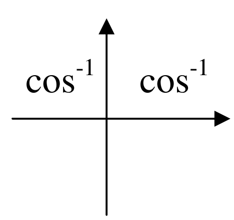
$\arctan\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ $\qquad$ $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ $\qquad$ $\arccos\left(\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

Why only one answer? Remember you calculators only returned values in certain quadrants when we were working with inverce functions? arcsin is only defined from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ inclusively, which are quadrants I and IV. arctan is also only defined from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, which are quadrants I and IV. However, arccos is defined from 0 to π inclusively, which are quadrants I and II.

This is reinforced when you use the inv tan, inv cos, or inv sin keys on you calculator.







Notice: For each function, one quadrant has positive values and the other quadrant has negative values.

Remember: An inverse trig function returns an angle. Given a trig value as *x* (a ratio), it returns a *y* that is an angle!

The Inverse Trigonometric Functions

Summary:
arcsin is defined in a range of $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and domain of $[-1,1]$
arctan is defined in a range of $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (Remember, at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, the tangent is undefined) and domain of all real numbers
arccos is defined in a range of $[0,\pi]$ and domain of $[-1,1]$

Find the exact value of the expression whenever it is defined.

$\sin^{-1}\left(-\frac{1}{2}\right)$ $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ $\tan^{-1}(-1)$

$\arcsin\left(\frac{1}{2}\right)$ $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ $\arctan\left(\sqrt{3}\right)$

For the following problems: **Always work from the inside out!!!**

$\sin\left[\arcsin\left(\frac{\sqrt{2}}{2}\right)\right]$ $\cos[\cos^{-1}(1)]$ $\tan[\tan^{-1}(5)]$

You will notice when the trig function is on the ‘outside’, the two functions ‘eliminate’ each other and the result is the trig value. **This may not happen when the inverse function is on the ‘outside’. Therefore, always work from the inside out.**

The Inverse Trigonometric Functions

$\arcsin\left(\sin\frac{5\pi}{4}\right)$ $\arccos\left(\cos\frac{5\pi}{4}\right)$ $\arctan\left(\tan\frac{5\pi}{4}\right)$

Notice: In the above 3 examples, the argument in each was $\frac{5\pi}{4}$. However, none of the answers were $\frac{5\pi}{4}$. It is outside of the range. Also, notice that the 3 answers are not the same.

$\sin\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ $\cos\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$ $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Find the exact value of the expression whenever it is defined.

$\sin\left[2\arccos\left(-\frac{5}{13}\right)\right]$

$=\sin[2\theta]$

$=2\sin\theta\cos\theta$

$\theta=\arccos\left(-\frac{5}{13}\right)$ T

$\therefore \cos\theta=-\frac{5}{13}$

The angle θ is in Q II.

$\cos\left[2\sin^{-1}\left(\frac{4}{5}\right)\right]$

$\theta=\sin^{-1}\left(\frac{4}{5}\right)$

$\therefore \sin\theta=\frac{4}{5}$

The angle θ is in Q I

The Inverse Trigonometric Functions

$$\tan\left[2\tan^{-1}\left(-\frac{9}{40}\right)\right]$$

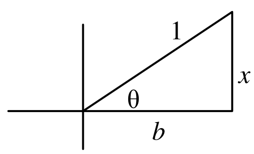
Write the expression as an algebraic expression in ***x* for *x* > 0.**

$$\cos\left(\sin^{-1} x\right)$$

$$\sin\theta = \frac{x}{1}$$

$$= \cos\theta$$

$$= \frac{\sqrt{1-x^2}}{1}$$



$$x^2 + b^2 = 1^2$$
$$b^2 = 1 - x^2$$
$$b = \sqrt{1-x^2}$$

$$= \sqrt{1-x^2}$$

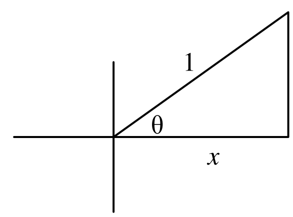
Proof (using a calculator)

$$\cos(\sin^{-1} x) = \sqrt{1-x^2}$$
Let $x = 0.26$
$$\cos(\sin^{-1}(0.26) = \sqrt{1-(0.26)^2}$$
$$\cos(0.263022203) = \sqrt{1-0.0676}$$
$$0.9656086 = \sqrt{0.9324}$$
$$0.9656086 = 0.9656086$$

$$\cos\theta = \frac{x}{1}$$

$$\sin\left(2\arccos x\right)$$

$$= \sin 2\theta$$



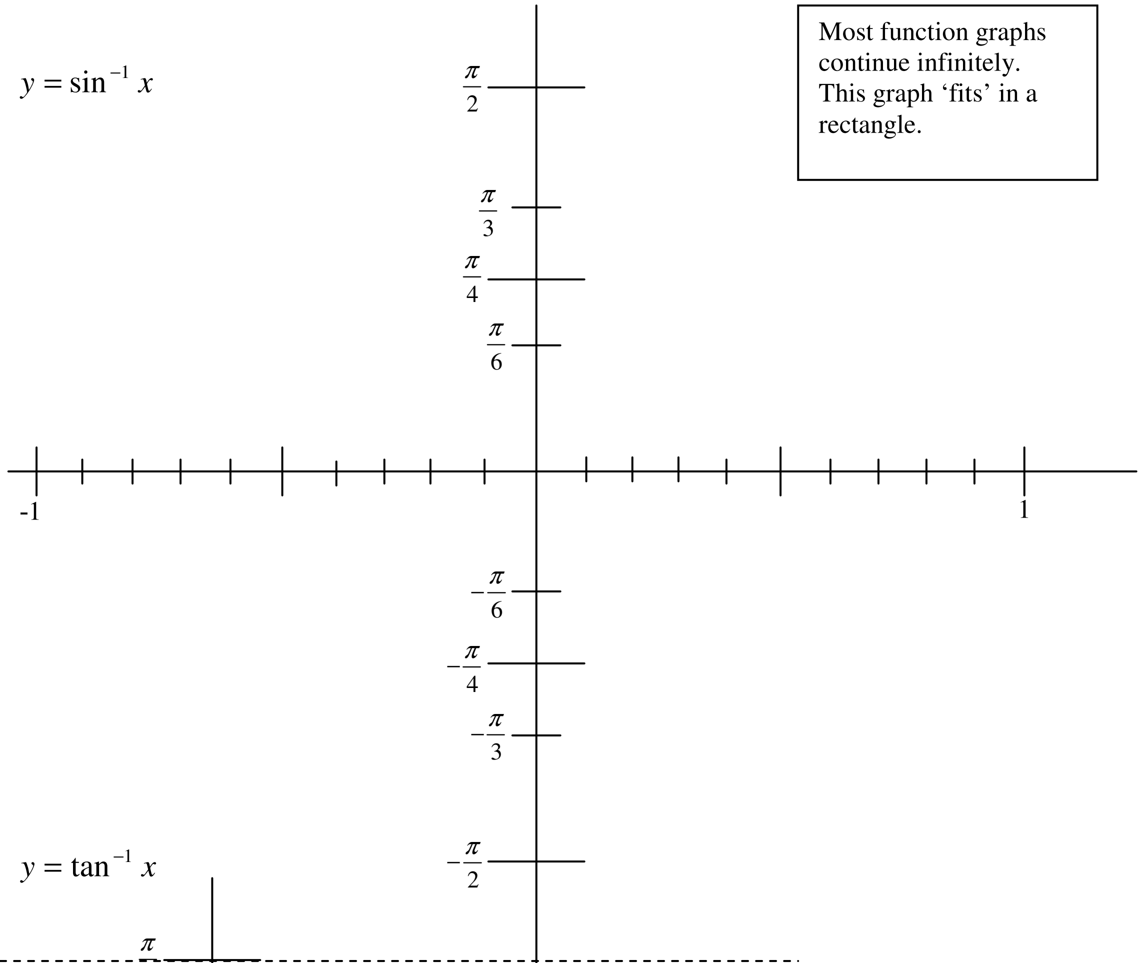
The Inverse Trigonometric Functions

Most function graphs continue infinitely. This graph 'fits' in a rectangle.

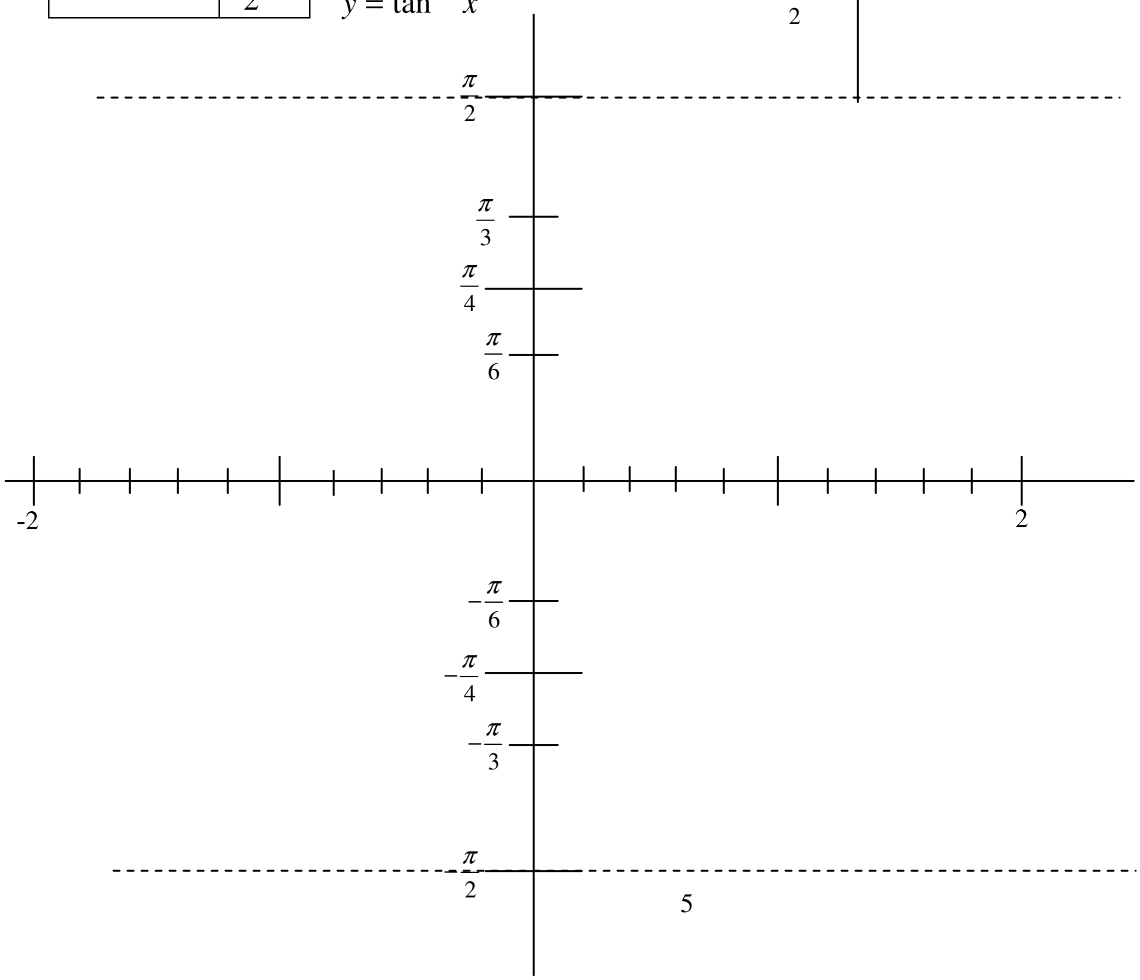
Graph:

Trig Value	Angle
-1	$-\frac{\pi}{2}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$
$-\frac{1}{\sqrt{2}}$	$-\frac{\pi}{4}$
$-\frac{1}{2}$	$-\frac{\pi}{6}$
0	0
$\frac{1}{2}$	$\frac{\pi}{6}$
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
1	$\frac{\pi}{2}$

$y = \sin^{-1} x$

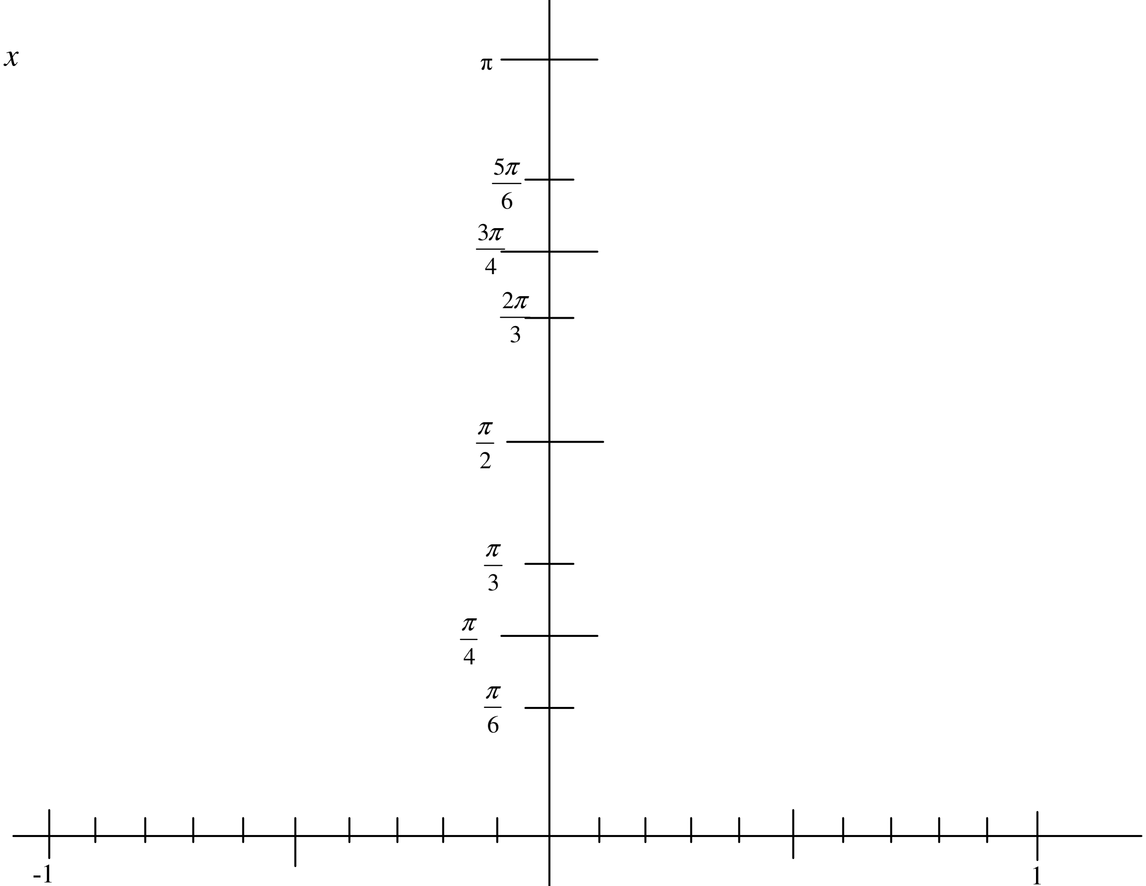


$y = \tan^{-1} x$

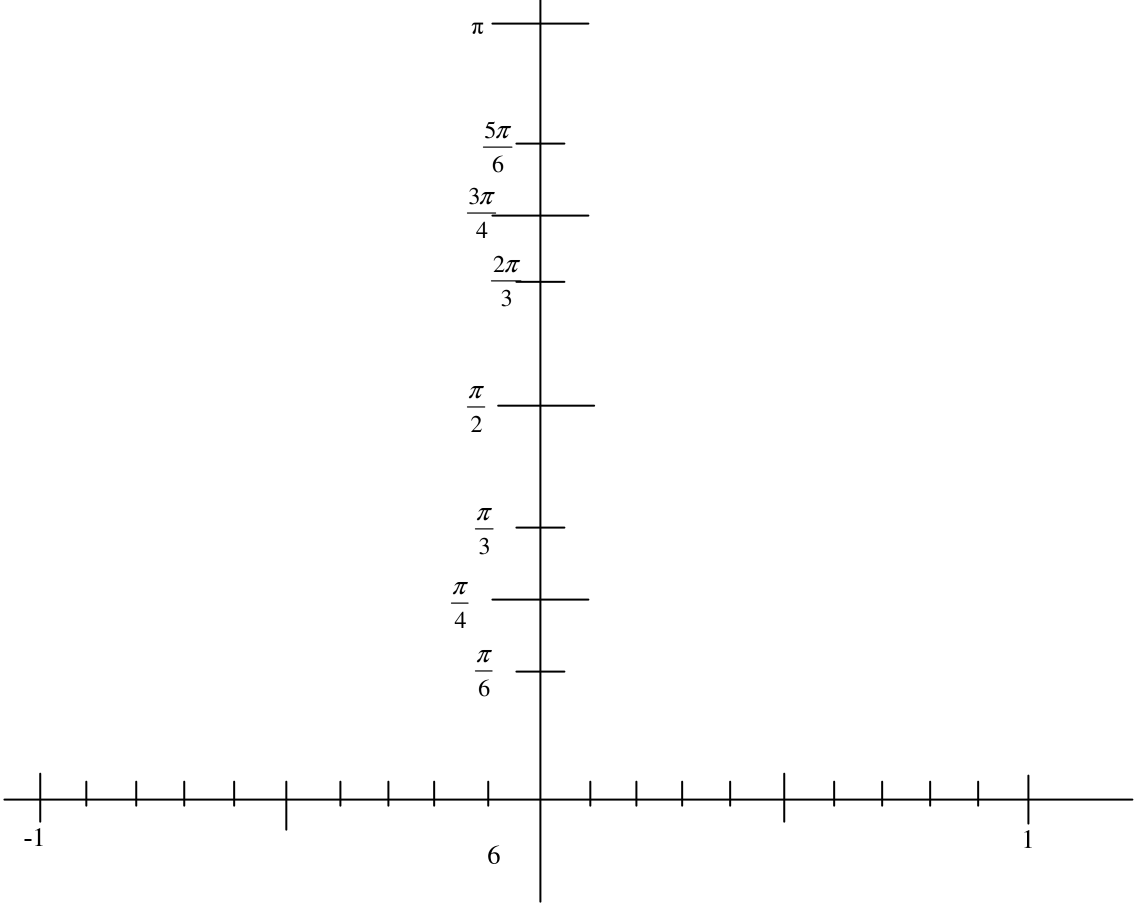


The Inverse Trigonometric Functions

$y = \cos^{-1} x$



$y = \frac{1}{2}\cos^{-1} x$



The Inverse Trigonometric Functions

$y = \cos^{-1}(2x)$