MA 220 Lesson 12 Notes
Section 3.4 of Calculus part of textbook

Tangent Line to a curve:
To understand the tangent line, we must first discuss a secant line. A secant line will intersect a curve at more than one point, where a tangent line only intersects a curve at one point and is an indication of the direction of the curve.

Figure 27 on page 162 of the calculus part of the textbook shows a tangent line to a curve. Figure 29 on page 163 shows a secant line to the curve $f(x)$ through points $R(a, f(a))$ and $S(a+h, f(a+h))$. The slope of the secant line would be $m = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$.

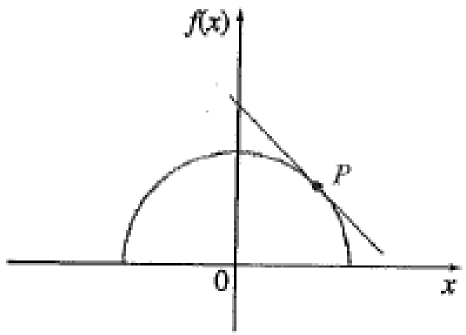


FIGURE 27

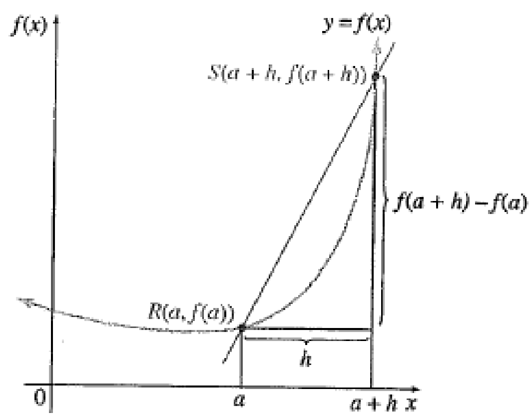
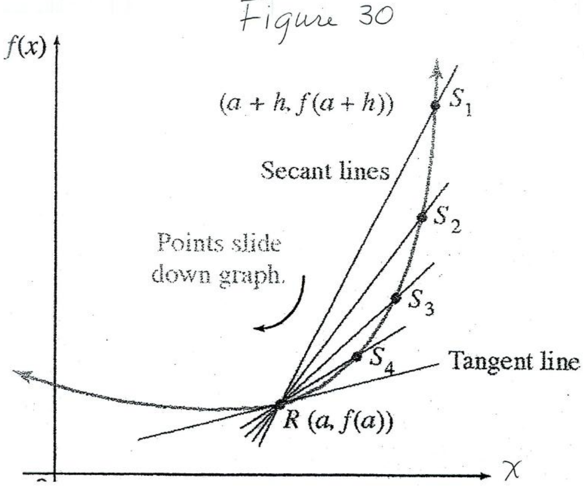


FIGURE 29

Figure 30



Imagine that points *R* and *S* in figure 29get closer and closer together. The closer the points become the closer the secant line would approximate the tangent line. As the value of *h* goes to 0 (*h* or run or the horizontal change), we would approximate the slope of the tangent line.

Slope of a Tangent Line to *y* = *f*(*x*) at a point (*a*, *f* (*a*)) is the following limit.

$$m = \lim_{h \to 0}\left(\frac{f(a+h) - f(a)}{h}\right)$$

Examples:

Ex 1: Find the equation of the secant line through the curve $y = f(x) = x^2 - x$ containing point $(1,1)$ and $(3,6)$.

$$m = \frac{6-1}{3-1} = \frac{5}{2}$$
$$y - 1 = \frac{5}{2}(x-1)$$
$$y - 1 = \frac{5}{2}x - \frac{5}{2}$$
$$y = \frac{5}{2}x - \frac{3}{2}$$

Ex 2: Now find the equations of the tangent lines to $y = f(x) = x^2 - x$ at the points $(1,0)$ and $(3,6)$.

$$\text{slope} = m = \lim_{h\to 0}\left(\frac{f(a+h) - f(a)}{h}\right) = \lim_{h\to 0}\left(\frac{[(a+h)^2 - (a+h)] - [a^2 - a]}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{(a^2 + 2ah + h^2 - a - h) - (a^2 - a)}{h}\right) = \lim_{h\to 0}\left(\frac{a^2 + 2ah + h^2 - a - h - a^2 + a}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{h(2a+h-1)}{h}\right) = \lim_{h\to 0}(2a + h - 1) = 2a + 0 - 1 = 2a - 1$$

For the tangent line at (1,0):

$$m = 2(1) - 1 = 1$$
equation of tangent line:
$$y - 0 = 1(x-1)$$
$$y = x - 1$$
$$\boxed{y = x - 1}$$

For the tangent line at (3, 6):

$$m = 2(3) - 1 = 5$$
equation of tangent line:
$$y - 6 = 5(x-3)$$
$$y - 6 = 5x - 15$$
$$\boxed{y = 5x - 9}$$

Definition of the Derivative:

The slope of a tangent line to a curve is the definition we use for a function called the **derivative.**

The **derivative** of function is defined as… $f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$, provided this limit exists. The notation $f'(x)$ means the derivative of function $f(x)$ with respect to *x* and is read '*f* prime of *x*'. We also can say that the function *f* is **differentiable** at *x* and the process that finds f' is called differentiation. The derivative of a function represents the ***instantaneous rate of change*** of the function $f(x)$ with respect to *x*. (Real life examples represented by instantaneous rate of change include marginal cost, marginal profit, and velocity.) As seen above, we also note that the derivative represents the slope of a tangent line to a function's graph.

Compare the following to distinguish between the difference quotient and the derivative.

Difference Quotient	Derivative
$\frac{f(x+h)-f(x)}{h}$	$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$
• Slope of the secant line • Average rate of Change • Average Velocity • Average rate of change in cost, revenue, or profit	• Slope of a tangent line • Instantaneous rate of Change • Instantaneous velocity • Marginal cost, revenue, or profit

Examples of finding derivatives: (Note: The textbook calls this a '4-step procedure'.)

__Ex. 3__: a) Using the definition of a derivative, find the derivative of this function, $f(x)=2x+6$.

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)=\lim_{h\to 0}\left(\frac{[2(x+h)+6]-(2x+6)}{h}\right)$$

$$=\lim_{h\to o}\left(\frac{2x+2h+6-2x-6}{h}\right)=\lim_{h\to 0}\left(\frac{2h}{n}\right)=\lim_{h\to 0}2=2$$

b) Find $f'(2)$ and $f'(-3)$. Answer: $f'(2)=2,\ f'(-3)=2$

__Ex. 4__: a) Using the definition of a derivative, find the derivative of this function, $y=g(x)=\frac{3}{x}$ or $g(x)=3x^{-1}$.

$$\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)=\lim_{h\to 0}\left(\frac{\frac{3}{x+h}-\frac{3}{x}}{h}\right)=\lim_{h\to 0}\left(\frac{\frac{3x-3(x+h)}{x(x+h)}}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{3x-3x-3h}{x(x+h)}\cdot\frac{1}{h}\right)=\lim_{h\to 0}\left(\frac{-3h}{xh(x+h)}\right)=\lim_{h\to 0}\left(\frac{-3}{x(x+h)}\right)$$

$$=\frac{-3}{x(x+0)}=\boxed{\frac{-3}{x^2}}$$

b) Find $g'(3)$ and $g'(0)$. Answer: $g'(3)=\frac{-3}{3^2}=-\frac{3}{9}=\boxed{-\frac{1}{3}}$, $g'(0)=\frac{-3}{0^2}$, which is $\boxed{\text{not defined.}}$

<u>Ex 5</u>: a) Using the definition of a derivative, find the derivative of this function, $y = 3x^2 - 2x + 1$.

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{[3(x+h)^2 - 2(x+h)+1]-(3x^2-2x+1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3(x^2+2xh+h^2)-2(x+h)+1-(3x^2-2x+1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3x^2+6xh+3h^2-2x-2h+1-3x^2+2x-1}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(6x+3h-2)}{h}\right) = \lim_{h\to 0}(6x+3h-2) = 6x+3(0)-2 = \boxed{6x-2}$$

b) Find $g'(4)$ and $g'(-1)$. Answer: $g'(4) = 6(4)-2 = \boxed{22,}$ $g'(-1) = 6(-1)-2 = \boxed{-8}$

There are other examples (examples 5 and 6) on pages 170 and 171 of the textbook.

Ex 6: Let's use the definition of a derivative to find the derivatives of these functions, then evaluate the derivative for x = 5 and x = -3.

$a)$ $f(x) = 9x^2 - 8x$

$b)$ $F(x) = \dfrac{1}{2x}$

Ex. 7: For the function below, find (a) the equation of the secant line through the point where x = -1 and x = 3 and (b) the equation of the tangent lines when x = -1 and x = 3.

$f(x) = 6 - x^2$

A derivative may not exist at every point of a function. For example, **a derivative does not exist at a point where the function is not defined. Derivatives cannot exist where there are 'corners' or 'sharp points' on a graph.**

There is a good picture in the textbook (Figure 43 on page 174) that summarizes the various ways that a derivative would not exist at a given point or a given *x* value.

A derivative only exists for a function *f* at *x* if all of these conditions are met.

1) *f* is continuous at *x* (no breaks, jumps, or gaps)
2) *f* has a graph that is a smooth curve at *x*
3) *f* does not have a vertical tangent line at *x*

A derivative does *not* exist for a function *f* at *x* whenever any of the following are true.

1) *f* is discontinuous at *x* (a break, jump, hole, or gap)
2) *f* has a sharp corner at *x*
3) *f* has a vertical tangent line at *x*
4) *f* is not defined at *x*

See figure 43 on page 174. The function represented in the figure is not differentiable at x_1 and x_2**,** because the graph has sharp corners at those points. The function is not differentiable at x_3 because the graph is discontinuous at that point. The function is not differentiable at x_4 because the graph is discontinuous at that point (in fact there is an asymptote, which means the function is not defined at x_4. The function is not differentiable at x_5 because there is a vertical tangent line at that point. The function is not differentiable at x_6 because the function is not defined for that value and the graph is discontinuous at that point.

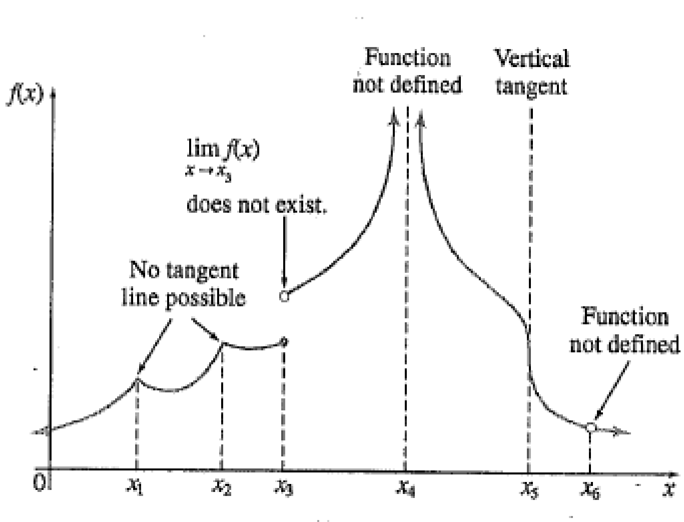
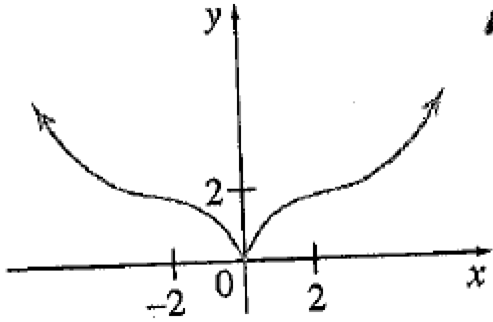


FIGURE 43

Ex. 8: Open your book to page 176 and look at problems 35, 36, 37, and 38. For each problem, at what value(s) of x (if any) does the function represented not have derivatives.

35.



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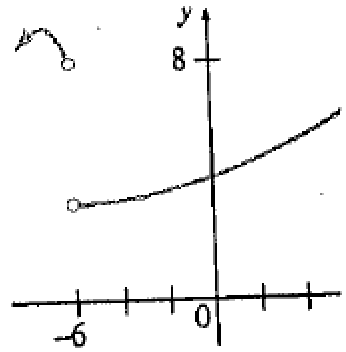

Number 35:

Number 36:

Number 37:

Number 38:

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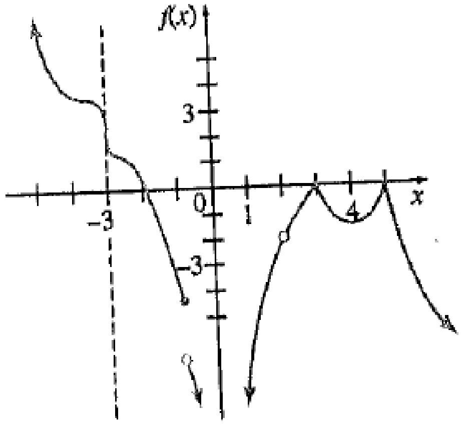

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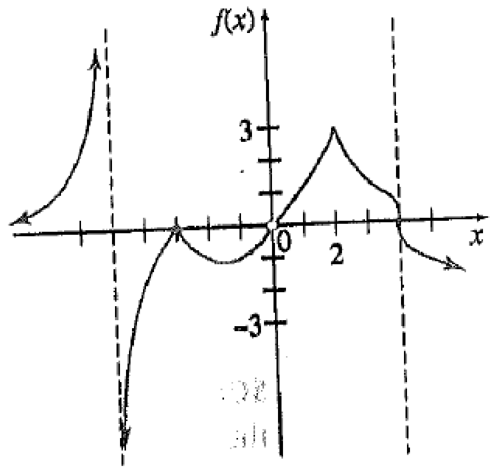

Ex. 9: The profit, in thousands of dollars, in a month for a company that spends <u>x thousand</u> dollars on advertising is given by $P(x) = 1000 + 32x - 2x^2$. Find the marginal profit for the following amount spent on advertising.

a) $8000
b) $6000
c) $12,000
d) $20,000

Ex 10: The cost in dollars to manufacture x graphing calculators is given by $C(x) = -0.005x^2 + 20x + 150$ when $0 \le x \le 2000$. Find the rate of change of cost with respect to the number manufactured when 100 calculators are made. When 1000 are made. (In other words, find the marginal cost.)

Ex 11: Find the equation of the tangent lines to the graph of $f(x) = 2x^2 - x + 3$ at $x = 0$ and $x = 2$.