where $u = f(y)$, and hence show that the general solution to Equation (1.8.26) is

$$y(x) = f^{-1}\left\{I^{-1}\left[\int I(x)q(x)\,dx + c\right]\right\},$$

where $I$ is given in (1.8.25), $f^{-1}$ is the inverse of $f$, and $c$ is an arbitrary constant.

**65.** Solve

$$\sec^2 y\,\frac{dy}{dx} + \frac{1}{2\sqrt{1+x}}\tan y = \frac{1}{2\sqrt{1+x}}.$$

## 1.9 Exact Differential Equations

For the next technique it is best to consider first-order differential equations written in differential form

$$M(x, y)\,dx + N(x, y)\,dy = 0, \tag{1.9.1}$$

where $M$ and $N$ are given functions, assumed to be sufficiently smooth.[8] The method that we will consider is based on the idea of a differential. Recall from a previous calculus course that if $\phi = \phi(x, y)$ is a function of two variables, $x$ and $y$, then the differential of $\phi$, denoted $d\phi$, is defined by

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy. \tag{1.9.2}$$

**Example 1.9.1** Solve

$$2x \sin y\,dx + x^2 \cos y\,dy = 0. \tag{1.9.3}$$

**Solution:** This equation is separable, but we will use a different technique to solve it. By inspection, we notice that

$$2x \sin y\,dx + x^2 \cos y\,dy = d(x^2 \sin y).$$

Consequently, Equation (1.9.3) can be written as $d(x^2 \sin y) = 0$, which implies that $x^2 \sin y$ is constant, hence the general solution to Equation (1.9.3) is

$$\sin y = \frac{c}{x^2},$$

where $c$ is an arbitrary constant. □

In the foregoing example we were able to write the given differential equation in the form $d\phi(x, y) = 0$, and hence obtain its solution. However, we cannot always do this. Indeed we see by comparing Equation (1.9.1) with (1.9.2) that the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

can be written as $d\phi = 0$ if and only if

$$M = \frac{\partial \phi}{\partial x} \quad \text{and} \quad N = \frac{\partial \phi}{\partial y}$$

for some function $\phi$. This motivates the following definition:

[8] This means we assume that the functions $M$ and $N$ have continuous derivatives of sufficiently high order.

**DEFINITION 1.9.2**

The differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be **exact** in a region $R$ of the $xy$-plane if there exists a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M, \qquad \frac{\partial \phi}{\partial y} = N, \tag{1.9.4}$$

for all $(x, y)$ in $R$.

Any function $\phi$ satisfying (1.9.4) is called a **potential function** for the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0.$$

We emphasize that if such a function exists, then the preceding differential equation can be written as

$$d\phi = 0.$$

This is why such a differential equation is called an exact differential equation. From the previous example, a potential function for the differential equation

$$2x \sin y\,dx + x^2 \cos y\,dy = 0$$

is

$$\phi(x, y) = x^2 \sin y.$$

We now show that if a differential equation is exact and we can find a potential function $\phi$, its solution can be written down immediately.

**Theorem 1.9.3** The general solution to an exact equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is defined implicitly by

$$\phi(x, y) = c,$$

where $\phi$ satisfies (1.9.4) and $c$ is an arbitrary constant.

**Proof** We rewrite the differential equation in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

Since the differential equation is exact, there exists a potential function $\phi$ (see (1.9.4)) such that

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0.$$

But this implies that $\partial\phi/\partial x = 0$. Consequently, $\phi(x, y)$ is a function of $y$ only. By a similar argument, which we leave to the reader, we can deduce that $\phi(x, y)$ is a function of $x$ only. We conclude therefore that $\phi(x, y) = c$, where $c$ is a constant. ■

## Remarks

1. The potential function $\phi$ is a function of two variables $x$ and $y$, and we interpret the relationship $\phi(x, y) = c$ as defining $y$ implicitly as a function of $x$. The preceding theorem states that this relationship defines the general solution to the differential equation for which $\phi$ is a potential function.
2. Geometrically, Theorem 1.9.3 says that the solution curves of an exact differential equation are the family of curves $\phi(x, y) = k$, where $k$ is a constant. These are called the **level curves** of the function $\phi(x, y)$.

The following two questions now arise:

1. How can we tell whether a given differential equation is exact?
2. If we have an exact equation, how do we find a potential function?

The answers are given in the next theorem and its proof.

**Theorem 1.9.4** **(Test for Exactness)** Let $M$, $N$, and their first partial derivatives $M_y$ and $N_x$, be continuous in a (simply connected[9]) region $R$ of the $xy$-plane. Then the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact for all $x$, $y$ in $R$ if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{1.9.5}$$

**Proof** We first prove that exactness implies the validity of Equation (1.9.5). If the differential equation is exact, then by definition there exists a potential function $\phi(x, y)$ such that $\phi_x = M$ and $\phi_y = N$. Thus, taking partial derivatives, $\phi_{xy} = M_y$ and $\phi_{yx} = N_x$. Since $M_y$ and $N_x$ are continuous in $R$, it follows that $\phi_{xy}$ and $\phi_{yx}$ are continuous in $R$. But, from multivariable calculus, this implies that $\phi_{xy} = \phi_{yx}$ and hence that $M_y = N_x$.

We now prove the converse. Thus we assume that Equation (1.9.5) holds and must prove that there exists a potential function $\phi$ such that

$$\frac{\partial \phi}{\partial x} = M \tag{1.9.6}$$

and

$$\frac{\partial \phi}{\partial y} = N. \tag{1.9.7}$$

The proof is constructional. That is, we will actually find a potential function $\phi$. We begin by integrating Equation (1.9.6) with respect to $x$, holding $y$ fixed (this is a partial integration) to obtain

$$\phi(x, y) = \int^x M(s, y)\,ds + h(y), \tag{1.9.8}$$

[9] Roughly speaking, simply connected means that the interior of any closed curve drawn in the region also lies in the region. For example, the interior of a circle is a simply connected region, although the region between two concentric circles is not.

where $h(y)$ is an arbitrary function of $y$ (this is the integration "constant" that we must allow to depend on $y$, since we held $y$ fixed in performing the integration[10]). We now show how to determine $h(y)$ so that the function $f$ defined in (1.9.8) also satisfies Equation (1.9.7). Differentiating (1.9.8) partially with respect to $y$ yields

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\int^x M(s, y)\,ds + \frac{dh}{dy}.$$

In order that $\phi$ satisfy Equation (1.9.7) we must choose $h(y)$ to satisfy

$$\frac{\partial}{\partial y}\int^x M(s, y)\,ds + \frac{dh}{dy} = N(x, y).$$

That is,

$$\frac{dh}{dy} = N(x, y) - \frac{\partial}{\partial y}\int^x M(s, y)\,ds. \tag{1.9.9}$$

Since the left-hand side of this expression is a function of $y$ only, we must show, for consistency, that the right-hand side also depends only on $y$. Taking the derivative of the right-hand side with respect to $x$ yields

$$\begin{aligned}\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int^x M(s, y)\,ds\right) &= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x \partial y}\int^x M(s, y)\,ds \\ &= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int^x M(s, y)\,ds\right) \\ &= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.\end{aligned}$$

Thus, using (1.9.5), we have

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int^x M(s, y)\,ds\right) = 0,$$

so that the right-hand side of Equation (1.9.9) does depend only on $y$. It follows that (1.9.9) is a consistent equation, and hence we can integrate both sides with respect to $y$ to obtain

$$h(y) = \int^y N(x, t)\,dt - \int^y \frac{\partial}{\partial t}\left(\int^x M(s, t)\,ds\right) dt.$$

Finally, substituting into (1.9.8) yields the potential function

$$\phi(x, y) = \int^x M(s, y)\,dx + \int^y N(x, t)\,dt - \int^y \frac{\partial}{\partial t}\left(\int^x M(s, t)\,ds\right) dt. \quad \blacksquare$$

**Remark** There is no need to memorize the final result for $\phi$. For each particular problem, one can construct an appropriate potential function from first principles. This is illustrated in Examples 1.9.6 and 1.9.7.

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[10]Throughout the text, $\int^x f(t)\,dt$ means "evaluate the indefinite integral $\int f(t)\,dt$ and replace $t$ with $x$ in the result."

**Example 1.9.5** Determine whether the given differential equation is exact.

1. $[1 + \ln(xy)]\, dx + (x/y)\, dy = 0$.
2. $x^2 y\, dx - (xy^2 + y^3)\, dy = 0$.

**Solution:**

1. In this case, $M = 1 + \ln(xy)$ and $N = x/y$, so that $M_y = 1/y = N_x$. It follows from the previous theorem that the differential equation is exact.
2. In this case, we have $M = x^2 y$, $N = -(xy^2 + y^3)$, so that $M_y = x^2$, whereas $N_x = -y^2$. Since $M_y \neq N_x$, the differential equation is not exact. □

**Example 1.9.6** Find the general solution to $2xe^y\, dx + (x^2 e^y + \cos y)\, dy = 0$.

**Solution:** We have

$$M(x, y) = 2xe^y, \qquad N(x, y) = x^2 e^y + \cos y,$$

so that

$$M_y = 2xe^y = N_x.$$

Hence the given differential equation is exact, and so there exists a potential function $\phi$ such that (see Definition 1.9.2)

$$\frac{\partial \phi}{\partial x} = 2xe^y, \tag{1.9.10}$$

$$\frac{\partial \phi}{\partial y} = x^2 e^y + \cos y. \tag{1.9.11}$$

Integrating Equation (1.9.10) with respect to $x$, holding $y$ fixed, yields

$$\phi(x, y) = x^2 e^y + h(y), \tag{1.9.12}$$

where $h$ is an arbitrary function of $y$. We now determine $h(y)$ such that (1.9.12) also satisfies Equation (1.9.11). Taking the derivative of (1.9.12) with respect to $y$ yields

$$\frac{\partial \phi}{\partial y} = x^2 e^y + \frac{dh}{dy}. \tag{1.9.13}$$

Equations (1.9.11) and (1.9.13) give two expressions for $\partial \phi / \partial y$. This allows us to determine $h$. Subtracting Equation (1.9.11) from Equation (1.9.13) gives the consistency requirement

$$\frac{dh}{dy} = \cos y,$$

which implies, upon integration, that

$$h(y) = \sin y,$$

where we have set the integration constant equal to zero without loss of generality, since we require only one potential function. Substitution into (1.9.12) yields the potential function

$$\phi(x, y) = x^2e^y + \sin y.$$

Consequently, the given differential equation can be written as

$$d(x^2e^y + \sin y) = 0,$$

and so, from Theorem 1.9.3, the general solution is

$$x^2e^y + \sin y = c. \qquad \square$$

Notice that the solution obtained in the preceding example is an implicit solution. Owing to the nature of the way in which the potential function for an exact equation is obtained, this is usually the case.

**Example 1.9.7** Find the general solution to

$$\left[\sin(xy) + xy\cos(xy) + 2x\right]dx + \left[x^2\cos(xy) + 2y\right]dy = 0.$$

**Solution:** We have

$$M(x, y) = \sin(xy) + xy\cos(xy) + 2x \quad \text{and} \quad N(x, y) = x^2\cos(xy) + 2y.$$

Thus,

$$M_y = 2x\cos(xy) - x^2y\sin(xy) = N_x,$$

and so the differential equation is exact. Hence there exists a potential function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = \sin(xy) + xy\cos(xy) + 2x, \qquad (1.9.14)$$

$$\frac{\partial \phi}{\partial y} = x^2\cos(xy) + 2y. \qquad (1.9.15)$$

In this case, Equation (1.9.15) is the simpler equation, and so we integrate it with respect to $y$, holding $x$ fixed, to obtain

$$\phi(x, y) = x\sin(xy) + y^2 + g(x), \qquad (1.9.16)$$

where $g(x)$ is an arbitrary function of $x$. We now determine $g(x)$, and hence $\phi$, from (1.9.14) and (1.9.16). Differentiating (1.9.16) partially with respect to $x$ yields

$$\frac{\partial \phi}{\partial x} = \sin(xy) + xy\cos(xy) + \frac{dg}{dx}. \qquad (1.9.17)$$

Equations (1.9.14) and (1.9.17) are consistent if and only if

$$\frac{dg}{dx} = 2x.$$

Hence, upon integrating,

$$g(x) = x^2,$$

where we have once more set the integration constant to zero without loss of generality, since we require only one potential function. Substituting into (1.9.16) gives the potential function

$$\phi(x, y) = x \sin xy + x^2 + y^2.$$

The original differential equation can therefore be written as

$$d(x \sin xy + x^2 + y^2) = 0,$$

and hence the general solution is

$$x \sin xy + x^2 + y^2 = c. \qquad \square$$

**Remark** At first sight the above procedure appears to be quite complicated. However, with a little bit of practice, the steps are seen to be, in fact, fairly straightforward. As we have shown in Theorem 1.9.4, the method works in general, provided one starts with an exact differential equation.

## Integrating Factors

Usually a given differential equation will not be exact. However, sometimes it is possible to multiply the differential equation by a nonzero function to obtain an exact equation that can then be solved using the technique we have described in this section. Notice that the solution to the resulting exact equation will be the same as that of the original equation, since we multiply by a nonzero function.

**DEFINITION 1.9.8**

A nonzero function $I(x, y)$ is called an **integrating factor** for the differential equation $M(x, y)dx + N(x, y)dy = 0$ if the differential equation

$$I(x, y)M(x, y)\, dx + I(x, y)N(x, y)\, dy = 0$$

is exact.

**Example 1.9.9** Show that $I = x^2y$ is an integrating factor for the differential equation

$$(3y^2 + 5x^2y)\, dx + (3xy + 2x^3)\, dy = 0. \qquad (1.9.18)$$

**Solution:** Multiplying the given differential equation (which is not exact) by $x^2y$ yields

$$(3x^2y^3 + 5x^4y^2)\, dx + (3x^3y^2 + 2x^5y)\, dy = 0. \qquad (1.9.19)$$

Thus,

$$M_y = 9x^2y^2 + 10x^4y = N_x,$$

so that the differential equation (1.9.19) is exact, and hence $I = x^2y$ is an integrating factor for Equation (1.9.18). Indeed we leave it as an exercise to verify that (1.9.19) can be written as

$$d(x^3y^3 + x^5y^2) = 0,$$

so that the general solution to Equation (1.9.19) (and hence the general solution to Equation (1.9.18)) is defined implicitly by

$$x^3y^3 + x^5y^2 = c.$$

That is,

$$x^3y^2(y + x^2) = c. \qquad \square$$

As shown in the next theorem, using the test for exactness, it is straightforward to determine the conditions that a function $I(x, y)$ must satisfy in order to be an integrating factor for the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$.

**Theorem 1.9.10** The function $I(x, y)$ is an integrating factor for

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1.9.20}$$

if and only if it is a solution to the partial differential equation

$$N\frac{\partial I}{\partial x} - M\frac{\partial I}{\partial y} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)I. \tag{1.9.21}$$

**Proof** Multiplying Equation (1.9.20) by $I$ yields

$$IM\,dx + IN\,dy = 0.$$

This equation is exact if and only if

$$\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial x}(IN),$$

that is, if and only if

$$\frac{\partial I}{\partial y}M + I\frac{\partial M}{\partial y} = \frac{\partial I}{\partial x}N + I\frac{\partial N}{\partial x}.$$

Rearranging the terms in this equation yields Equation (1.9.21). ■

The preceding theorem is not too useful in general, since it is usually no easier to solve the partial differential equation (1.9.21) to find $I$ than it is to solve the original Equation (1.9.20). However, it sometimes happens that an integrating factor exists that depends only on one variable. We now show that Theorem 1.9.10 can be used to determine when such an integrating factor exists and also to actually find a corresponding integrating factor.

**Theorem 1.9.11** Consider the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$.

1. There exists an integrating factor that is dependent only on $x$ if and only if $(M_y - N_x)/N = f(x)$, a function of $x$ only. In such a case, an integrating factor is

$$I(x) = e^{\int f(x)\,dx}.$$

2. There exists an integrating factor that is dependent only on $y$ if and only if $(M_y - N_x)/M = g(y)$, a function of $y$ only. In such a case, an integrating factor is

$$I(y) = e^{-\int g(y)\,dy}.$$

**Proof** For part 1 of the theorem, we begin by assuming that $I = I(x)$ is an integrating factor for $M(x, y)\,dx + N(x, y)\,dy = 0$. Then $\partial I/\partial y = 0$, and so, from (1.9.21), $I$ is a solution to

$$\frac{dI}{dx}N = (M_y - N_x)I.$$

That is,

$$\frac{1}{I}\frac{dI}{dx} = \frac{M_y - N_x}{N}.$$

Since, by assumption, $I$ is a function of $x$ only, it follows that the left-hand side of this expression depends only on $x$ and hence also the right-hand side.

Conversely, suppose that $(M_y - N_x)/N = f(x)$, a function of $x$ only. Then, dividing (1.9.21) by $N$, it follows that $I$ is an integrating factor for $M(x, y)\,dx + N(x, y)\,dy = 0$ if and only if it is a solution to

$$\frac{\partial I}{\partial x} - \frac{M}{N}\frac{\partial I}{\partial y} = If(x). \tag{1.9.22}$$

We must show that this differential equation has a solution $I$ that depends on $x$ only. We do this by explicitly integrating the differential equation under the assumption that $I = I(x)$. Indeed, if $I = I(x)$, then Equation (1.9.22) reduces to

$$\frac{dI}{dx} = If(x),$$

which is a separable equation with solution

$$I(x) = e^{\int f(x)\,dx}$$

The proof of part 2 is similar, and so we leave it as an exercise (see Problem 30 ). ■

**Example 1.9.12** Solve

$$(2x - y^2)\,dx + xy\,dy = 0, \qquad x > 0. \tag{1.9.23}$$

**Solution:** The equation is not exact ($M_y \neq N_x$). However,

$$\frac{M_y - N_x}{N} = \frac{-2y - y}{xy} = -\frac{3}{x},$$

which is a function of $x$ only. It follows from part 1 of the preceding theorem that an integrating factor for Equation (1.9.23) is

$$I(x) = e^{-\int (3/x)dx} = e^{-3\ln x} = x^{-3}.$$

Multiplying Equation (1.9.23) by $I$ yields the exact equation

$$(2x^{-2} - x^{-3}y^2)\,dx + x^{-2}y\,dy = 0. \qquad (1.9.24)$$

(The reader should check that this is exact, although it must be, by the previous theorem.) We leave it as an exercise to verify that a potential function for Equation (1.9.24) is

$$\phi(x, y) = \frac{1}{2}x^{-2}y^2 - 2x^{-1},$$

and hence the general solution to (1.9.23) is given implicitly by

$$\frac{1}{2}x^{-2}y^2 - 2x^{-1} = c,$$

or equivalently,

$$y^2 - 4x = c_1x^2. \qquad \square$$

## Exercises for 1.9

### Key Terms

Exact differential equation, Potential function, Integrating factor.

### Skills

- Be able to determine whether or not a given differential equation is exact.
- Given the partial derivatives $\partial\phi/\partial x$ and $\partial\phi/\partial y$ of a potential function $\phi(x, y)$, be able to determine $\phi(x, y)$.
- Be able to find the general solution to an exact differential equation.
- When circumstances allow, be able to use an integrating factor to convert a given differential equation into an exact differential equation with the same solution set.

### True-False Review

For Questions 1–9, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

**1.** The differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact in a simply connected region $R$ if $M_x$ and $N_y$ are continuous partial derivatives with $M_x = N_y$.

**2.** The solution to an exact differential equation is called a potential function.

**3.** If $M(x)$ and $N(y)$ are continuous functions, then the differential equation $M(x)\,dx + N(y)\,dy = 0$ is exact.

**4.** If $(M_y - N_x)/N(x, y)$ is a function of $x$ only, then the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ becomes exact when it is multiplied through by

$$I(x) = \exp\left(\int (M_y - N_x)/N(x, y)\,dx\right).$$

**5.** There is a unique potential function for an exact differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$.

**6.** The differential equation

$$(2ye^{2x} - \sin y)\,dx + (e^{2x} - x\cos y)\,dy = 0$$

is exact.

**7.** The differential equation

$$\frac{-2xy}{(x^2+y)^2}\,dx + \frac{x^2}{(x^2+y)^2}\,dy = 0$$

is exact.

**8.** The differential equation

$$(y^2 + \cos x)\,dx + 2xy^2\,dy = 0$$

is exact.

**9.** The differential equation

$$(e^{x \sin y} \sin y)\,dx + (e^{x \sin y} \cos y)\,dy = 0$$

is exact.

## Problems

For Problems 1–3, determine whether the given differential equation is exact.

**1.** $(y + 3x^2)\,dx + x dy = 0$.

**2.** $[\cos(xy) - xy\sin(xy)]\,dx - x^2 \sin(xy)\,dy = 0$.

**3.** $ye^{xy}\,dx + (2y - xe^{xy})\,dy = 0$.

For Problems 4–12, solve the given differential equation.

**4.** $2xy\,dx + (x^2 + 1)\,dy = 0$.

**5.** $(y^2 + \cos x)\,dx + (2xy + \sin y)\,dy = 0$.

**6.** $x^{-1}(xy - 1)\,dx + y^{-1}(xy + 1)\,dy = 0$.

**7.** $(4e^{2x} + 2xy - y^2)\,dx + (x - y)^2\,dy = 0$.

**8.** $(y^2 - 2x)\,dx + 2xy\,dy = 0$.

**9.** $\left(\frac{1}{x} - \frac{y}{x^2+y^2}\right) dx + \frac{x}{x^2+y^2}\,dy = 0$.

**10.** $[1 + \ln(xy)]\,dx + xy^{-1}\,dy = 0$.

**11.** $[y\cos(xy) - \sin x]\,dx + x\cos(xy)\,dy = 0$.

**12.** $(2xy + \cos y)\,dx + (x^2 - x\sin y - 2y)\,dy = 0$.

For Problems 13–15, solve the given initial-value problem.

**13.** $(3x^2 \ln x + x^2 - y)\,dx - x dy = 0, \quad y(1) = 5$.

**14.** $2x^2 y' + 4xy = 3\sin x, \quad y(2\pi) = 0$.

**15.** $(ye^{xy} + \cos x)\,dx + xe^{xy}\,dy = 0, \quad y(\pi/2) = 0$.

**16.** Show that if $\phi(x, y)$ is a potential function for $M(x, y)\,dx + N(x, y)\,dy = 0$, then so is $\phi(x, y) + c$, where $c$ is an arbitrary constant. This shows that potential functions are uniquely defined only up to an additive constant.

For Problems 17–19, determine whether the given function is an integrating factor for the given differential equation.

**17.** $I(x, y) = \cos(xy)$, $[\tan(xy) + xy]\,dx + x^2\,dy = 0$.

**18.** $I(x) = \sec x$, $[2x - (x^2 + y^2)\tan x]\,dx + 2y\,dy = 0$.

**19.** $I(x, y) = y^{-2}e^{-x/y}$, $y(x^2 - 2xy)\,dx - x^3\,dy = 0$.

For Problems 20–26, determine an integrating factor for the given differential equation, and hence find the general solution.

**20.** $(xy - 1)\,dx + x^2\,dy = 0$.

**21.** $y\,dx - (2x + y^4)\,dy = 0$.

**22.** $x^2 y\,dx + y(x^3 + e^{-3y} \sin y)\,dy = 0$.

**23.** $(y - x^2)\,dx + 2x dy = 0, \quad x > 0$.

**24.** $xy[2\ln(xy) + 1]\,dx + x^2\,dy = 0, \quad x > 0$.

**25.** $\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{1}{(1+x^2)^2}$.

**26.** $(3xy - 2y^{-1})\,dx + x(x + y^{-2})\,dy = 0$.

For Problems 27–29, determine the values of the constants $r$ and $s$ such that $I(x, y) = x^r y^s$ is an integrating factor for the given differential equation.

**27.** $(y^{-1} - x^{-1})\,dx + (xy^{-2} - 2y^{-1})\,dy = 0$.

**28.** $y(5xy^2 + 4)\,dx + x(xy^2 - 1)\,dy = 0$.

**29.** $2y(y + 2x^2)\,dx + x(4y + 3x^2)\,dy = 0$.

**30.** Prove that if $(M_y - N_x)/M = g(y)$, a function of $y$ only, then an integrating factor for

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is $I(y) = e^{-\int g(y)\,dy}$.

**31.** Consider the general first-order *linear* differential equation

$$\frac{dy}{dx} + p(x)y = q(x), \tag{1.9.25}$$

where $p(x)$ and $q(x)$ are continuous functions on some interval $(a, b)$.

**(a)** Rewrite Equation (1.9.25) in differential form, and show that an integrating factor for the resulting equation is

$$I(x) = e^{\int p(x)dx}. \tag{1.9.26}$$

**(b)** Show that the general solution to Equation (1.9.25) can be written in the form

$$y(x) = I^{-1}\left\{\int^x I(t)q(t)\,dt + c\right\},$$

where $I$ is given in Equation (1.9.26), and $c$ is an arbitrary constant.

## 1.10 Numerical Solution to First-Order Differential Equations

So far in this chapter we have investigated first-order differential equations geometrically via slope fields, and analytically by trying to construct exact solutions to certain types of differential equations. Certainly, for most first-order differential equations, it simply is not possible to find analytic solutions, since they will not fall into the few classes for which solution techniques are available. Our final approach to analyzing first-order differential equations is to look at the possibility of constructing a numerical approximation to the unique solution to the initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \tag{1.10.1}$$

We consider three techniques that give varying levels of accuracy. In each case, we generate a sequence of approximations $y_1, y_2, \ldots$ to the value of the exact solution at the points $x_1, x_2, \ldots,$ where $x_{n+1} = x_n + h, n = 0, 1, \ldots,$ and $h$ is a real number. We emphasize that numerical methods do not generate a formula for the solution to the differential equation. Rather they generate a sequence of approximations to the value of the solution at specified points. Furthermore, if we use a sufficient number of points, then by plotting the points $(x_i, y_i)$ and joining them with straight-line segments, we are able to obtain an overall approximation to the solution curve corresponding to the solution of the given initial-value problem. This is how the approximate solution curves were generated in the preceding sections via the computer algebra system Maple. There are many subtle ideas associated with constructing numerical solutions to initial-value problems that are beyond the scope of this text. Indeed, a full discussion of the application of numerical methods to differential equations is best left for a future course in numerical analysis.

### Euler's Method

Suppose we wish to approximate the solution to the initial-value problem (1.10.1) at $x = x_1 = x_0 + h$, where $h$ is small. The idea behind Euler's method is to use the tangent line to the solution curve through $(x_0, y_0)$ to obtain such an approximation. (See Figure 1.10.1.)

The equation of the tangent line through $(x_0, y_0)$ is

$$y(x) = y_0 + m(x - x_0),$$

where $m$ is the slope of the curve at $(x_0, y_0)$. From Equation (1.10.1), $m = f(x_0, y_0)$, so

$$y(x) = y_0 + f(x_0, y_0)(x - x_0).$$