20. $\begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

21. $\begin{bmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \\ 6 & -6 & 12 \end{bmatrix}$.

22. $\begin{bmatrix} 3 & 5 & -12 \\ 2 & 3 & -7 \\ -2 & -1 & 1 \end{bmatrix}$.

23. $\begin{bmatrix} 1 & -1 & -1 & 2 \\ 3 & -2 & 0 & 7 \\ 2 & -1 & 2 & 4 \\ 4 & -2 & 3 & 8 \end{bmatrix}$.

24. $\begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & -6 & 2 & 7 \\ 4 & -8 & 3 & 10 \end{bmatrix}$.

25. $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$.

Many forms of technology have commands for performing elementary row operations on a matrix A. For example, in the linear algebra package of Maple, the three elementary row operations are

- swaprow(A, i, j): permute rows i and j
- mulrow(A, i, k): multiply row i by k
- addrow(A, i, j): add k times row i to row j

◊ For Problems 26–28, use some form of technology to determine a row-echelon form of the given matrix.

26. The matrix in Problem 14.

27. The matrix in Problem 15.

28. The matrix in Problem 18.

◊ Many forms of technology also have built-in functions for directly determining the reduced row-echelon form of a given matrix A. For example, in the linear algebra package of Maple, the appropriate command is rref(A). In Problems 29–31, use technology to determine directly the reduced row-echelon form of the given matrix.

29. The matrix in Problem 21.

30. The matrix in Problem 24.

31. The matrix in Problem 25.

2.5 Gaussian Elimination

We now illustrate how elementary row-operations applied to the augmented matrix of a system of linear equations can be used first to determine whether the system is consistent, and second, if the system is consistent, to find all of its solutions. In doing so, we will develop the general theory for linear systems of equations.

Example 2.5.1 Determine the solution set to

$$\begin{aligned} 3x_1 - 2x_2 + 2x_3 &= 9, \\ x_1 - 2x_2 + x_3 &= 5, \\ 2x_1 - x_2 - 2x_3 &= -1. \end{aligned} \qquad (2.5.1)$$

Solution: We first use elementary row operations to reduce the augmented matrix of the system to row-echelon form.

$$\begin{bmatrix} 3 & -2 & 2 & 9 \\ 1 & -2 & 1 & 5 \\ 2 & -1 & -2 & -1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 3 & -2 & 2 & 9 \\ 2 & -1 & -2 & -1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 4 & -1 & -6 \\ 0 & 3 & -4 & -11 \end{bmatrix}$$

$$\overset{3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 3 & -4 & -11 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -13 & -26 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-3)$, $A_{13}(-2)$ **3.** $A_{32}(-1)$ **4.** $A_{23}(-3)$ **5.** $M_3(-1/13)$

The system corresponding to this row-echelon form of the augmented matrix is

$$x_1 - 2x_2 + x_3 = 5, \quad (2.5.2)$$
$$x_2 + 3x_3 = 5, \quad (2.5.3)$$
$$x_3 = 2, \quad (2.5.4)$$

which can be solved by *back substitution*. From Equation (2.5.4), $x_3 = 2$. Substituting into Equation (2.5.3) and solving for x_2, we find that $x_2 = -1$. Finally, substituting into Equation (2.5.2) for x_3 and x_2 and solving for x_1 yields $x_1 = 1$. Thus, our original system of equations has the unique solution $(1, -1, 2)$, and the solution set to the system is

$$S = \{(1, -1, 2)\},$$

which is a subset of $\mathbb{R}^3$. □

The process of reducing the augmented matrix to row-echelon form and then using back substitution to solve the equivalent system is called **Gaussian elimination**. The particular case of Gaussian elimination that arises when the augmented matrix is reduced to reduced row-echelon form is called **Gauss-Jordan elimination**.

Example 2.5.2 Use Gauss-Jordan elimination to determine the solution set to

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1,\\ 2x_1 + 5x_2 - x_3 &= 3,\\ x_1 + 3x_2 + 2x_3 &= 6. \end{aligned}$$

Solution: In this case, we first reduce the augmented matrix of the system to reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

1. $A_{12}(-2)$, $A_{13}(-1)$ **2.** $A_{21}(-2)$, $A_{23}(-1)$ **3.** $M_3(1/2)$ **4.** $A_{31}(3)$, $A_{32}(-1)$

The augmented matrix is now in reduced row-echelon form. The equivalent system is

$$\begin{aligned} x_1 \quad &= 5,\\ x_2 \quad &= -1,\\ x_3 &= 2. \end{aligned}$$

and the solution can be read off directly as $(5, -1, 2)$. Consequently, the given system has solution set

$$S = \{(5, -1, 2)\}$$

in $\mathbb{R}^3$. □

We see from the preceding two examples that the advantage of Gauss-Jordan elimination over Gaussian elimination is that it does not require back substitution. However, the disadvantage is that reducing the augmented matrix to reduced row-echelon form requires more elementary row operations than reduction to row-echelon form. It can be shown, in fact, that in general, Gaussian elimination is the more computationally efficient technique. As we will see in the next section, the main reason for introducing the Gauss-Jordan method is its application to the computation of the inverse of an $n \times n$ matrix.

Remark The Gaussian elimination method is so systematic that it can be programmed easily on a computer. Indeed, many large-scale programs for solving linear systems are based on the row-reduction method.

In both of the preceding examples,

$$\text{rank}(A) = \text{rank}(A^{\#}) = \text{number of unknowns in the system}$$

and the system had a unique solution. More generally, we have the following lemma:

Lemma 2.5.3 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^{\#}$ denote the augmented matrix of the system. If $\text{rank}(A) = \text{rank}(A^{\#}) = n$, then the system has a unique solution.

Proof If $\text{rank}(A) = \text{rank}(A^{\#}) = n$, then there are n leading ones in any row-echelon form of A, hence back substitution gives a unique solution. The form of the row-echelon form of $A^{\#}$ is shown below, with $m - n$ rows of zeros at the bottom of the matrix omitted and where the $*$'s denote unknown elements of the row-echelon form.

$$\begin{bmatrix} 1 & * & * & * & \dots & * & * \\ 0 & 1 & * & * & \dots & * & * \\ 0 & 0 & 1 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & * \end{bmatrix}$$ ■

Note that $\text{rank}(A)$ cannot exceed $\text{rank}(A^{\#})$. Thus, there are only two possibilities for the relationship between $\text{rank}(A)$ and $\text{rank}(A^{\#})$: $\text{rank}(A) < \text{rank}(A^{\#})$ or $\text{rank}(A) = \text{rank}(A^{\#})$. We now consider what happens in these cases.

Example 2.5.4 Determine the solution set to

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 1, \\ 2x_1 + 3x_2 + x_3 \quad\quad &= 4, \\ 3x_1 + 5x_2 + 3x_3 - x_4 &= 5. \end{aligned}$$

Solution: We use elementary row operations to reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 2 & 6 & -4 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

1. $A_{12}(-2)$, $A_{13}(-3)$ **2.** $A_{23}(-2)$

The last row tells us that the system of equations has no solution (that is, it is inconsistent), since it requires

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -2,$$

which is clearly impossible. The solution set to the system is thus the empty set $\emptyset$. □

In the previous example, $\text{rank}(A) = 2$, whereas $\text{rank}(A^{\#}) = 3$. Thus, $\text{rank}(A) < \text{rank}(A^{\#})$, and the corresponding system has no solution. Next we establish that this result is true in general.

Lemma 2.5.5 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^{\#}$ denote the augmented matrix of the system. If $\text{rank}(A) < \text{rank}(A^{\#})$, then the system is inconsistent.

Proof If $\text{rank}(A) < \text{rank}(A^{\#})$, then there will be one row in the reduced row-echelon form of the augmented matrix whose first nonzero element arises in the last column. Such a row corresponds to an equation of the form

$$0x_1 + 0x_2 + \cdots + 0x_n = 1,$$

which has no solution. Consequently, the system is inconsistent. ■

Finally, we consider the case when $\text{rank}(A) = \text{rank}(A^{\#})$. If $\text{rank}(A) = n$, we have already seen in Lemma 2.5.3 that the system has a unique solution. We now consider an example in which $\text{rank}(A) < n$.

Example 2.5.6 Determine the solution set to

$$\begin{aligned} 5x_1 - 6x_2 + x_3 &= 4, \\ 2x_1 - 3x_2 + x_3 &= 1, \\ 4x_1 - 3x_2 - x_3 &= 5. \end{aligned} \tag{2.5.5}$$

Solution: We begin by reducing the augmented matrix of the system.

$$\begin{bmatrix} 5 & -6 & 1 & 4 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 3 & -3 & 3 \\ 0 & 9 & -9 & 9 \end{bmatrix}$$
$$\overset{3}{\sim} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 9 & -9 & 9 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. $A_{31}(-1)$ **2.** $A_{12}(-2)$, $A_{13}(-4)$ **3.** $M_2(1/3)$ **4.** $A_{23}(-9)$

The augmented matrix is now in row-echelon form, and the equivalent system is

$$x_1 - 3x_2 + 2x_3 = -1, \tag{2.5.6}$$
$$x_2 - x_3 = 1. \tag{2.5.7}$$

Since we have three variables, but only two equations relating them, we are free to specify one of the variables arbitrarily. The variable that we choose to specify is called a **free variable** or **free parameter**. The remaining variables are then determined by the system of equations and are called **bound variables** or **bound parameters**. In the foregoing system, we take x_3 as the free variable and set

$$x_3 = t,$$

where t can assume any real value[5]. It follows from (2.5.7) that

$$x_2 = 1 + t.$$

[5] When considering systems of equations with complex coefficients, we allow free variables to assume *complex* values as well.

Further, from Equation (2.5.6),

$$x_1 = -1 + 3(1 + t) - 2t = 2 + t.$$

Thus the solution set to the given system of equations is the following subset of $\mathbb{R}^3$:

$$S = \{(2 + t, 1 + t, t) : t \in \mathbb{R}\}.$$

The system has an infinite number of solutions, obtained by allowing the parameter t to assume all real values. For example, two particular solutions of the system are

$$(2, 1, 0) \qquad \text{and} \qquad (0, -1, -2),$$

corresponding to $t = 0$ and $t = -2$, respectively. Note that we can also write the solution set S above in the form

$$S = \{(2, 1, 0) + t(1, 1, 1) : t \in \mathbb{R}\}.$$ □

Remark The geometry of the foregoing solution is as follows. The given system (2.5.5) can be interpreted as consisting of three planes in 3-space. Any solution to the system gives the coordinates of a point of intersection of the three planes. In the preceding example the planes intersect in a line whose parametric equations are

$$x_1 = 2 + t, \qquad x_2 = 1 + t, \qquad x_3 = t.$$

(See Figure 2.3.1.)

In general, the solution to a consistent $m \times n$ system of linear equations may involve more than one free variable. Indeed, the number of free variables will depend on how many nonzero rows arise in any row-echelon form of the augmented matrix, $A^\#$, of the system; that is, it will depend on the rank of $A^\#$. More precisely, if $\text{rank}(A^\#) = r^\#$, then the equivalent system will have only $r^\#$ relationships between the n variables. Consequently, provided the system is consistent,

$$\text{number of free variables} = n - r^\#.$$

We therefore have the following lemma.

Lemma 2.5.7 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^\#$ denote the augmented matrix of the system and let $r^\# = \text{rank}(A^\#)$. If $r^\# = \text{rank}(A) < n$, then the system has an infinite number of solutions, indexed by $n - r^\#$ free variables.

Proof As discussed before, any row-echelon equivalent system will have only $r^\#$ equations involving the n variables, and so there will be $n - r^\# > 0$ free variables. If we assign arbitrary values to these free variables, then the remaining $r^\#$ variables will be uniquely determined, by back substitution, from the system. Since each free variable can assume infinitely many values, in this case there are an infinite number of solutions to the system. ■

Example 2.5.8 Use Gaussian elimination to solve

$$\begin{aligned} x_1 - 2x_2 + 2x_3 - x_4 &= 3, \\ 3x_1 + x_2 + 6x_3 + 11x_4 &= 16, \\ 2x_1 - x_2 + 4x_3 + 4x_4 &= 9. \end{aligned}$$

Solution: A row-echelon form of the augmented matrix of the system is

$$\begin{bmatrix} 1 & -2 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so that we have two free variables. The equivalent system is

$$x_1 - 2x_2 + 2x_3 - x_4 = 3, \tag{2.5.8}$$

$$x_2 + 2x_4 = 1. \tag{2.5.9}$$

Notice that we cannot choose any two variables freely. For example, from Equation (2.5.9), we cannot specify both x_2 and x_4 independently. The bound variables should be taken as those that correspond to leading 1s in the row-echelon form of $A^{\#}$, since these are the variables that can always be determined by back substitution (they appear as the leftmost variable in some equation of the system corresponding to the row echelon form of the augmented matrix).

> Choose as free variables those variables that **do not** correspond to a leading 1 in a row-echelon form of $A^{\#}$.

Applying this rule to Equations (2.5.8) and (2.5.9), we choose x_3 and x_4 as free variables and therefore set

$$x_3 = s, \quad x_4 = t.$$

It then follows from Equation (2.5.9) that

$$x_2 = 1 - 2t.$$

Substitution into (2.5.8) yields

$$x_1 = 5 - 2s - 3t,$$

so that the solution set to the given system is the following subset of $\mathbb{R}^4$:

$$\begin{aligned} S &= \{(5 - 2s - 3t, 1 - 2t, s, t) : s, t \in \mathbb{R}\}. \\ &= \{(5, 1, 0, 0) + s(-2, 0, 1, 0) + t(-3, -2, 0, 1) : s, t \in \mathbb{R}\}. \end{aligned}$$ □

Lemmas 2.5.3, 2.5.5, and 2.5.7 completely characterize the solution properties of an $m \times n$ linear system. Combining the results of these three lemmas gives the next theorem.

Theorem 2.5.9 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let r denote the rank of A, and let $r^{\#}$ denote the rank of the augmented matrix of the system. Then

1. If $r < r^{\#}$, the system is inconsistent.

2. If $r = r^{\#}$, the system is consistent and

  (a) There exists a unique solution if and only if $r^{\#} = n$.

  (b) There exists an infinite number of solutions if and only if $r^{\#} < n$.

Homogeneous Linear Systems

Many problems that we will meet in the future will require the solution to a homogeneous system of linear equations. The general form for such a system is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0, \end{aligned} \tag{2.5.10}$$

or, in matrix form, $A\mathbf{x} = \mathbf{0}$, where A is the coefficient matrix of the system and $\mathbf{0}$ denotes the m-vector whose elements are all zeros.

Corollary 2.5.10 The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is consistent for any coefficient matrix A, with a solution given by $\mathbf{x} = \mathbf{0}$.

Proof We can see immediately from (2.5.10) that if $\mathbf{x} = \mathbf{0}$, then $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} = \mathbf{0}$ is a solution to the homogeneous linear system.

Alternatively, we can deduce the consistency of this system from Theorem 2.5.9 as follows. The augmented matrix $A^{\#}$ of a homogeneous linear system differs from that of the coefficient matrix A only by the addition of a column of zeros, a feature that does not affect the rank of the matrix. Consequently, for a homogeneous system, we have $\text{rank}(A^{\#}) = \text{rank}(A)$, and therefore, from Theorem 2.5.9, such a system is necessarily consistent. ■

Remarks

1. The solution $\mathbf{x} = \mathbf{0}$ is referred to as the **trivial solution**. Consequently, from Theorem 2.5.9, a homogeneous system either has *only* the trivial solution or has an infinite number of solutions (one of which must be the trivial solution).
2. Once more it is worth mentioning the geometric interpretation of Corollary 2.5.10 in the case of a homogeneous system with three unknowns. We can regard each equation of such a system as defining a plane. Owing to the homogeneity, each plane passes through the origin, hence the planes intersect at least at the origin.

Often we will be interested in determining whether a given homogeneous system has an infinite number of solutions, and not in actually obtaining the solutions. The following corollary to Theorem 2.5.9 can sometimes be used to determine by inspection whether a given homogeneous system has nontrivial solutions:

Corollary 2.5.11 A homogeneous system of m linear equations in n unknowns, with $m < n$, has an infinite number of solutions.

Proof Let r and $r^{\#}$ be as in Theorem 2.5.9. Using the fact that $r = r^{\#}$ for a homogeneous system, we see that since $r^{\#} \leq m < n$, Theorem 2.5.9 implies that the system has an infinite number of solutions. ■

Remark If $m \geq n$, then we may or may not have nontrivial solutions, depending on whether the rank of the augmented matrix, $r^{\#}$, satisfies $r^{\#} < n$ or $r^{\#} = n$, respectively. We encourage the reader to construct linear systems that illustrate each of these two possibilities.

Example 2.5.12 Determine the solution set to $A\mathbf{x} = \mathbf{0}$, if $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 3 & 7 \end{bmatrix}$.

Solution: The augmented matrix of the system is

$$\begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 7 & 0 \end{bmatrix},$$

with reduced row-echelon form

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equivalent system is

$$x_2 = 0,$$
$$x_3 = 0.$$

It is tempting, but incorrect, to conclude from this that the solution to the system is $x_1 = x_2 = x_3 = 0$. Since x_1 does not occur in the system, it is a free variable and therefore *not necessarily* zero. Consequently, the correct solution to the foregoing system is $(r, 0, 0)$, where r is a free variable, and the solution set is $\{(r, 0, 0) : r \in \mathbb{R}\}$. □

The linear systems that we have so far encountered have all had real coefficients, and we have considered corresponding real solutions. The techniques that we have developed for solving linear systems are also applicable to the case when our system has complex coefficients. The corresponding solutions will also be complex.

Remark In general, the simplest method of putting a leading 1 in a position that contains the complex number $a + ib$ is to multiply the corresponding row by the scalar $\left(\frac{1}{a^2+b^2}\right)(a - ib)$. This is illustrated in steps 1 and 4 in the next example. If difficulties are encountered, consultation of Appendix A is in order.

Example 2.5.13 Determine the solution set to

$$\begin{aligned} (1+2i)x_1 + 4x_2 + (3+i)x_3 &= 0, \\ (2-i)x_1 + (1+i)x_2 + 3x_3 &= 0, \\ 5ix_1 + (7-i)x_2 + (3+2i)x_3 &= 0. \end{aligned}$$

Solution: We reduce the augmented matrix of the system.

$$\begin{bmatrix} 1+2i & 4 & 3+i & 0 \\ 2-i & 1+i & 3 & 0 \\ 5i & 7-i & 3+2i & 0 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & \frac{4}{5}(1-2i) & 1-i & 0 \\ 2-i & 1+i & 3 & 0 \\ 5i & 7-i & 3+2i & 0 \end{bmatrix}$$

$$\overset{2}{\sim} \begin{bmatrix} 1 & \frac{4}{5}(1-2i) & 1-i & 0 \\ 0 & (1+i) - \frac{4}{5}(1-2i)(2-i) & 3-(1-i)(2-i) & 0 \\ 0 & (7-i) - 4i(1-2i) & (3+2i) - 5i(1-i) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{4}{5}(1-2i) & 1-i & 0 \\ 0 & 1+5i & 2+3i & 0 \\ 0 & -1-5i & -2-3i & 0 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & \frac{4}{5}(1-2i) & 1-i & 0 \\ 0 & 1+5i & 2+3i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & \frac{4}{5}(1-2i) & 1-i & 0 \\ 0 & 1 & \frac{1}{26}(17-7i) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. $M_1((1-2i)/5)$ **2.** $A_{12}(-(2-i))$, $A_{13}(-5i)$ **3.** $A_{23}(1)$ **4.** $M_2((1-5i)/26)$

This matrix is now in row-echelon form. The equivalent system is

$$\begin{aligned} x_1 + \tfrac{4}{5}(1-2i)x_2 + (1-i)x_3 &= 0, \\ x_2 + \tfrac{1}{26}(17-7i)x_3 &= 0. \end{aligned}$$

There is one free variable, which we take to be $x_3 = t$, where t can assume any *complex* value. Applying back substitution yields

$$\begin{aligned} x_2 &= \tfrac{1}{26}t(-17+7i) \\ x_1 &= -\tfrac{2}{65}t(1-2i)(-17+7i) - t(1-i) \\ &= -\tfrac{1}{65}t(59+17i) \end{aligned}$$

so that the solution set to the system is the subset of $\mathbb{C}^3$

$$\left\{\left(-\tfrac{1}{65}t(59+17i), \tfrac{1}{26}t(-17+7i), t\right) : t \in \mathbb{C}\right\}. \quad \square$$

Exercises for 2.5

Key Terms

Gaussian elimination, Gauss-Jordan elimination, Free variables, Bound (or leading) variables, Trivial solution.

Skills

- Be able to solve a linear system of equations by Gaussian elimination and by Gauss-Jordan elimination.
- Be able to identify free variables and bound variables and know how they are used to construct the solution set to a linear system.
- Understand the relationship between the ranks of A and $A^{\#}$, and how this affects the number of solutions to a linear system.

True-False Review

For Questions 1–6, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. The process by which a matrix is brought via elementary row operations to row-echelon form is known as Gauss-Jordan elimination.

2. A homogeneous linear system of equations is always consistent.

3. For a linear system $A\mathbf{x} = \mathbf{b}$, every column of the row-echelon form of A corresponds to either a bound variable or a free variable, but not both, of the linear system.

4. A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the last column of the row-echelon form of the augmented matrix $[A \;\; \mathbf{b}]$ is not a pivot column.

5. A linear system is consistent if and only if there are free variables in the row-echelon form of the corresponding augmented matrix.

6. The columns of the row-echelon form of $A^{\#}$ that contain the leading 1s correspond to the free variables.

Problems

For Problems 1–9, use Gaussian elimination to determine the solution set to the given system.

1.
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1,\\ 3x_1 + 5x_2 + x_3 &= 3,\\ 2x_1 + 6x_2 + 7x_3 &= 1. \end{aligned}$$

2.
$$\begin{aligned} 3x_1 - x_2 &= 1,\\ 2x_1 + x_2 + 5x_3 &= 4,\\ 7x_1 - 5x_2 - 8x_3 &= -3. \end{aligned}$$

3.
$$\begin{aligned} 3x_1 + 5x_2 - x_3 &= 14,\\ x_1 + 2x_2 + x_3 &= 3,\\ 2x_1 + 5x_2 + 6x_3 &= 2. \end{aligned}$$

4.
$$\begin{aligned} 6x_1 - 3x_2 + 3x_3 &= 12,\\ 2x_1 - x_2 + x_3 &= 4,\\ -4x_1 + 2x_2 - 2x_3 &= -8. \end{aligned}$$

5.
$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 14,\\ 3x_1 + x_2 - 2x_3 &= -1,\\ 7x_1 + 2x_2 - 3x_3 &= 3,\\ 5x_1 - x_2 - 2x_3 &= 5. \end{aligned}$$

6.
$$\begin{aligned} 2x_1 - x_2 - 4x_3 &= 5,\\ 3x_1 + 2x_2 - 5x_3 &= 8,\\ 5x_1 + 6x_2 - 6x_3 &= 20,\\ x_1 + x_2 - 3x_3 &= -3. \end{aligned}$$

7.
$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 1,\\ 2x_1 + 4x_2 - 2x_3 + 2x_4 &= 2,\\ 5x_1 + 10x_2 - 5x_3 + 5x_4 &= 5. \end{aligned}$$

8.
$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 1,\\ 2x_1 - 3x_2 + x_3 - x_4 &= 2,\\ x_1 - 5x_2 + 2x_3 - 2x_4 &= 1,\\ 4x_1 + x_2 - x_3 + x_4 &= 3. \end{aligned}$$

9.
$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 - 2x_5 &= 3,\\ x_3 + 4x_4 - 3x_5 &= 2,\\ 2x_1 + 4x_2 - x_3 - 10x_4 + 5x_5 &= 0. \end{aligned}$$

For Problems 10–15, use Gauss-Jordan elimination to determine the solution set to the given system.

10.
$$\begin{aligned} 2x_1 - x_2 - x_3 &= 2,\\ 4x_1 + 3x_2 - 2x_3 &= -1,\\ x_1 + 4x_2 + x_3 &= 4. \end{aligned}$$

11.
$$\begin{aligned} 3x_1 + x_2 + 5x_3 &= 2,\\ x_1 + x_2 - x_3 &= 1,\\ 2x_1 + x_2 + 2x_3 &= 3. \end{aligned}$$

12.
$$\begin{aligned} x_1 - 2x_3 &= -3,\\ 3x_1 - 2x_2 - 4x_3 &= -9,\\ x_1 - 4x_2 + 2x_3 &= -3. \end{aligned}$$

13.
$$\begin{aligned} 2x_1 - x_2 + 3x_3 - x_4 &= 3,\\ 3x_1 + 2x_2 + x_3 - 5x_4 &= -6,\\ x_1 - 2x_2 + 3x_3 + x_4 &= 6. \end{aligned}$$

14.
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 4,\\ x_1 - x_2 - x_3 - x_4 &= 2,\\ x_1 + x_2 - x_3 + x_4 &= -2,\\ x_1 - x_2 + x_3 + x_4 &= -8. \end{aligned}$$

15.
$$\begin{aligned} 2x_1 - x_2 + 3x_3 + x_4 - x_5 &= 11,\\ x_1 - 3x_2 - 2x_3 - x_4 - 2x_5 &= 2,\\ 3x_1 + x_2 - 2x_3 - x_4 + x_5 &= -2,\\ x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 &= -3,\\ 5x_1 - 3x_2 - 3x_3 + x_4 + 2x_5 &= 2. \end{aligned}$$

For Problems 16–20, determine the solution set to the system $A\mathbf{x} = \mathbf{b}$ for the given coefficient matrix A and right-hand side vector $\mathbf{b}$.

16. $A = \begin{bmatrix} 1 & -3 & 1 \\ 5 & -4 & 1 \\ 2 & 4 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 15 \\ -4 \end{bmatrix}.$

17. $A = \begin{bmatrix} 1 & 0 & 5 \\ 3 & -2 & 11 \\ 2 & -2 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$

18. $A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 8 \\ 5 \end{bmatrix}.$

19. $A = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & 1 & 3 & 7 \\ 3 & -2 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$.

20. $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 3 & 1 & -2 & 3 \\ 2 & 3 & 1 & 2 \\ -2 & 3 & 5 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 8 \\ 3 \\ -9 \end{bmatrix}$.

21. Determine all values of the constant k for which the following system has **(a)** no solution, **(b)** an infinite number of solutions, and **(c)** a unique solution.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3, \\ 2x_1 + 5x_2 + x_3 &= 7, \\ x_1 + x_2 - k^2x_3 &= -k. \end{aligned}$$

22. Determine all values of the constant k for which the following system has **(a)** no solution, **(b)** an infinite number of solutions, and **(c)** a unique solution.

$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 &= 0, \\ x_1 + x_2 + x_3 - x_4 &= 0, \\ 4x_1 + 2x_2 - x_3 + x_4 &= 0, \\ 3x_1 - x_2 + x_3 + kx_4 &= 0. \end{aligned}$$

23. Determine all values of the constants a and b for which the following system has **(a)** no solution, **(b)** an infinite number of solutions, and **(c)** a unique solution.

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4, \\ 3x_1 + 5x_2 - 4x_3 &= 16, \\ 2x_1 + 3x_2 - ax_3 &= b. \end{aligned}$$

24. Determine all values of the constants a and b for which the following system has **(a)** no solution, **(b)** an infinite number of solutions, and **(c)** a unique solution.

$$\begin{aligned} x_1 - ax_2 &= 3, \\ 2x_1 + x_2 &= 6, \\ -3x_1 + (a+b)x_2 &= 1. \end{aligned}$$

25. Show that the system

$$\begin{aligned} x_1 + x_2 + x_3 &= y_1, \\ 2x_1 + 3x_2 + x_3 &= y_2, \\ 3x_1 + 5x_2 + x_3 &= y_3, \end{aligned}$$

has an infinite number of solutions, provided that (y_1, y_2, y_3) lies on the plane whose equation is $y_1 - 2y_2 + y_3 = 0$.

26. Consider the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1, \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

Define Δ, Δ_1, and Δ_2 by

$$\Delta = a_{11}a_{22} - a_{12}a_{21},$$
$$\Delta_1 = a_{22}b_1 - a_{12}b_2, \qquad \Delta_2 = a_{11}b_2 - a_{12}b_1.$$

(a) Show that the given system has a unique solution if and only if $\Delta \neq 0$, and that the unique solution in this case is $x_1 = \Delta_1/\Delta,\ x_2 = \Delta_2/\Delta$.

(b) If $\Delta = 0$ and $a_{11} \neq 0$, determine the conditions on Δ_2 that would guarantee that the system has **(i)** no solution, **(ii)** an infinite number of solutions.

(c) Interpret your results in terms of intersections of straight lines.

Gaussian elimination with *partial pivoting* uses the following algorithm to reduce the augmented matrix:

1. Start with augmented matrix $A^\#$.
2. Determine the leftmost nonzero column.
3. Permute rows to put the element of largest absolute value in the pivot position.
4. Use elementary row operations to put zeros beneath the pivot position.
5. If there are no more nonzero rows below the pivot position, go to 7, otherwise go to 6.
6. Apply (2)–(5) to the submatrix consisting of the rows that lie below the pivot position.
7. The matrix is in reduced form.[6]

In Problems 27–30, use the preceding algorithm to reduce $A^\#$ and then apply back substitution to solve the equivalent system. Technology might be useful in performing the required row operations.

27. The system in Problem 1.

28. The system in Problem 5.

29. The system in Problem 6.

30. The system in Problem 10.

[6] Notice that this reduced form is *not* a row-echelon matrix.

31. **(a)** An $n \times n$ system of linear equations whose matrix of coefficients is a lower triangular matrix is called a **lower triangular system**. Assuming that $a_{ii} \neq 0$ for each i, devise a method for solving such a system that is analogous to the back-substitution method.

(b) Use your method from (a) to solve

$$\begin{aligned} x_1 \quad\quad\quad\quad &= 2, \\ 2x_1 - 3x_2 \quad\quad &= 1, \\ 3x_1 + x_2 - x_3 &= 8. \end{aligned}$$

32. Find all solutions to the following nonlinear system of equations:

$$\begin{aligned} 4x_1^3 + 2x_2^2 + 3x_3 &= 12, \\ x_1^3 - x_2^2 + x_3 &= 2, \\ 3x_1^3 + x_2^2 - x_3 &= 2. \end{aligned}$$

Does your answer contradict Theorem 2.5.9? Explain.

For Problems 33–43, determine the solution set to the given system.

33. $$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 0, \\ 2x_1 + x_2 + x_3 &= 0, \\ 5x_1 - 4x_2 + x_3 &= 0. \end{aligned}$$

34. $$\begin{aligned} 2x_1 + x_2 - x_3 &= 0, \\ 3x_1 - x_2 + 2x_3 &= 0, \\ x_1 - x_2 - x_3 &= 0, \\ 5x_1 + 2x_2 - 2x_3 &= 0. \end{aligned}$$

35. $$\begin{aligned} 2x_1 - x_2 - x_3 &= 0, \\ 5x_1 - x_2 + 2x_3 &= 0, \\ x_1 + x_2 + 4x_3 &= 0. \end{aligned}$$

36. $$\begin{aligned} (1+2i)x_1 + (1-i)x_2 + x_3 &= 0, \\ ix_1 + (1+i)x_2 - ix_3 &= 0, \\ 2ix_1 + x_2 + (1+3i)x_3 &= 0. \end{aligned}$$

37. $$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 0, \\ 6x_1 - x_2 + 2x_3 &= 0, \\ 12x_1 + 6x_2 + 4x_3 &= 0. \end{aligned}$$

38. $$\begin{aligned} 2x_1 + x_2 - 8x_3 &= 0, \\ 3x_1 - 2x_2 - 5x_3 &= 0, \\ 5x_1 - 6x_2 - 3x_3 &= 0, \\ 3x_1 - 5x_2 + x_3 &= 0. \end{aligned}$$

39. $$\begin{aligned} x_1 + (1+i)x_2 + (1-i)x_3 &= 0, \\ ix_1 + x_2 + ix_3 &= 0, \\ (1-2i)x_1 - (1-i)x_2 + (1-3i)x_3 &= 0. \end{aligned}$$

40. $$\begin{aligned} x_1 - x_2 + x_3 &= 0, \\ 3x_2 + 2x_3 &= 0, \\ 3x_1 - x_3 &= 0, \\ 5x_1 + x_2 - x_3 &= 0. \end{aligned}$$

41. $$\begin{aligned} 2x_1 - 4x_2 + 6x_3 &= 0, \\ 3x_1 - 6x_2 + 9x_3 &= 0, \\ x_1 - 2x_2 + 3x_3 &= 0, \\ 5x_1 - 10x_2 + 15x_3 &= 0. \end{aligned}$$

42. $$\begin{aligned} 4x_1 - 2x_2 - x_3 - x_4 &= 0, \\ 3x_1 + x_2 - 2x_3 + 3x_4 &= 0, \\ 5x_1 - x_2 - 2x_3 + x_4 &= 0. \end{aligned}$$

43. $$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 &= 0, \\ x_1 + x_2 + x_3 - x_4 &= 0, \\ 3x_1 - x_2 + x_3 - 2x_4 &= 0, \\ 4x_1 + 2x_2 - x_3 + x_4 &= 0. \end{aligned}$$

For Problems 44–54, determine the solution set to the system $A\mathbf{x} = \mathbf{0}$ for the given matrix A.

44. $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

45. $A = \begin{bmatrix} 1-i & 2i \\ 1+i & -2 \end{bmatrix}$.

46. $A = \begin{bmatrix} 1+i & 1-2i \\ -1+i & 2+i \end{bmatrix}$.

47. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

48. $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 2 & 2 \end{bmatrix}$.

49. $A = \begin{bmatrix} 2-3i & 1+i & i-1 \\ 3+2i & -1+i & -1-i \\ 5-i & 2i & -2 \end{bmatrix}$.

50. $A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -3 & 0 \\ 1 & 4 & 0 \end{bmatrix}$.

51. $A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & -1 & 7 \\ 2 & 1 & 8 \\ 1 & 1 & 5 \\ -1 & 1 & -1 \end{bmatrix}$.

52. $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 3 & -2 & 0 & 5 \\ -1 & 2 & 0 & 1 \end{bmatrix}$.

53. $A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 0 & -9 & 0 \\ -2 & 0 & 6 & 0 \end{bmatrix}$.

54. $A = \begin{bmatrix} 2+i & i & 3-2i \\ i & 1-i & 4+3i \\ 3-i & 1+i & 1+5i \end{bmatrix}$.

2.6 The Inverse of a Square Matrix

In this section we investigate the situation when, for a given $n \times n$ matrix A, there exists a matrix B satisfying

$$AB = I_n \quad \text{and} \quad BA = I_n \tag{2.6.1}$$

and derive an efficient method for determining B (when it does exist). As a possible application of the existence of such a matrix B, consider the $n \times n$ linear system

$$A\mathbf{x} = \mathbf{b}. \tag{2.6.2}$$

Premultiplying both sides of (2.6.2) by an $n \times n$ matrix B yields

$$(BA)\mathbf{x} = B\mathbf{b}.$$

Assuming that $BA = I_n$, this reduces to

$$\mathbf{x} = B\mathbf{b}. \tag{2.6.3}$$

Thus, we have determined a solution to the system (2.6.2) by a matrix multiplication. Of course, this depends on the existence of a matrix B satisfying (2.6.1), and even if such a matrix B does exist, it will turn out that using (2.6.3) to solve $n \times n$ systems is not very efficient computationally. Therefore it is generally not used in practice to solve $n \times n$ systems. However, from a theoretical point of view, a formula such as (2.6.3) is very useful. We begin the investigation by establishing that there can be at most one matrix B satisfying (2.6.1) for a given $n \times n$ matrix A.

Theorem 2.6.1 Let A be an $n \times n$ matrix. Suppose B and C are both $n \times n$ matrices satisfying

$$AB = BA = I_n, \tag{2.6.4}$$
$$AC = CA = I_n, \tag{2.6.5}$$

respectively. Then $B = C$.

Proof From (2.6.4), it follows that

$$C = CI_n = C(AB).$$

That is,

$$C = (CA)B = I_nB = B,$$

where we have used (2.6.5) to replace CA by I_n in the second step. ■

Since the identity matrix I_n plays the role of the number 1 in the multiplication of matrices, the properties given in (2.6.1) are the analogs for matrices of the properties

$$xx^{-1} = 1, \quad x^{-1}x = 1,$$

which holds for all (nonzero) numbers x. It is therefore natural to denote the matrix B in (2.6.1) by A^{-1} and to call it the inverse of A. The following definition introduces the appropriate terminology.