53. $A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 0 & -9 & 0 \\ -2 & 0 & 6 & 0 \end{bmatrix}$.

54. $A = \begin{bmatrix} 2+i & i & 3-2i \\ i & 1-i & 4+3i \\ 3-i & 1+i & 1+5i \end{bmatrix}$.

2.6 The Inverse of a Square Matrix

In this section we investigate the situation when, for a given $n \times n$ matrix A, there exists a matrix B satisfying

$$AB = I_n \quad \text{and} \quad BA = I_n \tag{2.6.1}$$

and derive an efficient method for determining B (when it does exist). As a possible application of the existence of such a matrix B, consider the $n \times n$ linear system

$$A\mathbf{x} = \mathbf{b}. \tag{2.6.2}$$

Premultiplying both sides of (2.6.2) by an $n \times n$ matrix B yields

$$(BA)\mathbf{x} = B\mathbf{b}.$$

Assuming that $BA = I_n$, this reduces to

$$\mathbf{x} = B\mathbf{b}. \tag{2.6.3}$$

Thus, we have determined a solution to the system (2.6.2) by a matrix multiplication. Of course, this depends on the existence of a matrix B satisfying (2.6.1), and even if such a matrix B does exist, it will turn out that using (2.6.3) to solve $n \times n$ systems is not very efficient computationally. Therefore it is generally not used in practice to solve $n \times n$ systems. However, from a theoretical point of view, a formula such as (2.6.3) is very useful. We begin the investigation by establishing that there can be at most one matrix B satisfying (2.6.1) for a given $n \times n$ matrix A.

Theorem 2.6.1 Let A be an $n \times n$ matrix. Suppose B and C are both $n \times n$ matrices satisfying

$$AB = BA = I_n, \tag{2.6.4}$$
$$AC = CA = I_n, \tag{2.6.5}$$

respectively. Then $B = C$.

Proof From (2.6.4), it follows that

$$C = CI_n = C(AB).$$

That is,

$$C = (CA)B = I_nB = B,$$

where we have used (2.6.5) to replace CA by I_n in the second step. ■

Since the identity matrix I_n plays the role of the number 1 in the multiplication of matrices, the properties given in (2.6.1) are the analogs for matrices of the properties

$$xx^{-1} = 1, \quad x^{-1}x = 1,$$

which holds for all (nonzero) numbers x. It is therefore natural to denote the matrix B in (2.6.1) by A^{-1} and to call it the inverse of A. The following definition introduces the appropriate terminology.

DEFINITION 2.6.2

Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} satisfying

$$AA^{-1} = A^{-1}A = I_n,$$

then we call A^{-1} *the* matrix **inverse** to A, or just *the* inverse of A. We say that A is **invertible** if A^{-1} exists.

Invertible matrices are sometimes called **nonsingular**, while matrices that are not invertible are sometimes called **singular**.

Remark It is important to realize that A^{-1} denotes the matrix that satisfies

$$AA^{-1} = A^{-1}A = I_n.$$

It does *not* mean $1/A$, which has no meaning whatsoever.

Example 2.6.3 If $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$, verify that $B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ is the inverse of A.

Solution: By direct multiplication, we find that

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

and

$$BA = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Consequently, (2.6.1) is satisfied, hence B is indeed the inverse of A. We therefore write

$$A^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$ □

We now return to the $n \times n$ system of Equations (2.6.2).

Theorem 2.6.4 If A^{-1} exists, then the $n \times n$ system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

has the *unique* solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

for every $\mathbf{b}$ in $\mathbb{R}^n$.

Proof We can verify by direct substitution that $\mathbf{x} = A^{-1}\mathbf{b}$ is indeed a solution to the linear system. The uniqueness of this solution is contained in the calculation leading from (2.6.2) to (2.6.3). ■

Our next theorem establishes when A^{-1} exists, and it also uncovers an efficient method for computing A^{-1}.

Theorem 2.6.5 An $n \times n$ matrix A is invertible if and only if $\text{rank}(A) = n$.

Proof If A^{-1} exists, then by Theorem 2.6.4, any $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution. Hence, Theorem 2.5.9 implies that $\text{rank}(A) = n$.

Conversely, suppose $\text{rank}(A) = n$. We must establish that there exists an $n \times n$ matrix X satisfying

$$AX = I_n = XA.$$

Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denote the column vectors of the identity matrix I_n. Since $\text{rank}(A) = n$, Theorem 2.5.9 implies that each of the linear systems

$$A\mathbf{x}_i = \mathbf{e}_i, \qquad i = 1, 2, \dots, n \tag{2.6.6}$$

has a unique solution[7] $\mathbf{x}_i$. Consequently, if we let $X = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$, where $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are the unique solutions of the systems in (2.6.6), then

$$A[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n] = [A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_n] = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n];$$

that is,

$$AX = I_n. \tag{2.6.7}$$

We must also show that, for the same matrix X,

$$XA = I_n.$$

Postmultiplying both sides of (2.6.7) by A yields

$$(AX)A = A.$$

That is,

$$A(XA - I_n) = 0_n. \tag{2.6.8}$$

Now let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ denote the column vectors of the $n \times n$ matrix $XA - I_n$. Equating corresponding column vectors on either side of (2.6.8) implies that

$$A\mathbf{y}_i = \mathbf{0}, \qquad i = 1, 2, \dots, n. \tag{2.6.9}$$

But, by assumption, $\text{rank}(A) = n$, and so each system in (2.6.9) has a unique solution that, since the systems are homogeneous, must be the trivial solution. Consequently, each $\mathbf{y}_i$ is the zero vector, and thus

$$XA - I_n = 0_n.$$

Therefore,

$$XA = I_n. \tag{2.6.10}$$

[7]Notice that for an $n \times n$ system $A\mathbf{x} = \mathbf{b}$, if $\text{rank}(A) = n$, then $\text{rank}(A^{\#}) = n$.

Equations (2.6.7) and (2.6.10) imply that $X = A^{-1}$. ■

We now have the following converse to Theorem 2.6.4.

Corollary 2.6.6 Let A be an $n \times n$ matrix. If $A\mathbf{x} = \mathbf{b}$ has a unique solution for some column n-vector $\mathbf{b}$, then A^{-1} exists.

Proof If $A\mathbf{x} = \mathbf{b}$ has a unique solution, then from Theorem 2.5.9, $\text{rank}(A) = n$, and so from the previous theorem, A^{-1} exists. ■

Remark In particular, the above corollary tells us that if the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, then A^{-1} exists.

Other criteria for deciding whether or not an $n \times n$ matrix A has an inverse will be developed in the next three chapters, but our goal at present is to develop a method for finding A^{-1}, should it exist.

Assuming that $\text{rank}(A) = n$, let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ denote the column vectors of A^{-1}. Then, from (2.6.6), these column vectors can be obtained by solving each of the $n \times n$ systems

$$A\mathbf{x}_i = \mathbf{e}_i, \qquad i = 1, 2, \dots, n.$$

As we now show, some computation can be saved if we employ the Gauss-Jordan method in solving these systems. We first illustrate the method when $n = 3$. In this case, from (2.6.6), the column vectors of A^{-1} are determined by solving the three linear systems

$$A\mathbf{x}_1 = \mathbf{e}_1, \qquad A\mathbf{x}_2 = \mathbf{e}_2, \qquad A\mathbf{x}_3 = \mathbf{e}_3.$$

The augmented matrices of these systems can be written as

$$\left[\begin{array}{c|c} \mathbf{A} & \begin{matrix}1\\0\\0\end{matrix}\end{array}\right], \left[\begin{array}{c|c} \mathbf{A} & \begin{matrix}0\\1\\0\end{matrix}\end{array}\right], \left[\begin{array}{c|c} \mathbf{A} & \begin{matrix}0\\0\\1\end{matrix}\end{array}\right],$$

respectively. Furthermore, since $\text{rank}(A) = 3$ by assumption, the reduced row-echelon form of A is I_3. Consequently, using elementary row operations to reduce the augmented matrix of the first system to reduced row-echelon form will yield, schematically,

$$\left[\begin{array}{c|c} \mathbf{A} & \begin{matrix}1\\0\\0\end{matrix}\end{array}\right] \sim \overset{\text{ERO}}{\cdots} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \end{array}\right],$$

which implies that the first column vector of A^{-1} is

$$\mathbf{x}_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Similarly, for the second system, the reduction

$$\left[\begin{array}{c|c} \mathbf{A} & \begin{matrix}0\\1\\0\end{matrix}\end{array}\right] \sim \overset{\text{ERO}}{\cdots} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

implies that the second column vector of A^{-1} is

$$\mathbf{x}_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Finally, for the third system, the reduction

$$\left[\begin{array}{c|c} \mathbf{A} & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \end{array}\right] \sim \overset{\text{ERO}}{\cdots} \sim \begin{bmatrix} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{bmatrix}$$

implies that the third column vector of A^{-1} is

$$\mathbf{x}_3 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Consequently,

$$A^{-1} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

The key point to notice is that in solving for $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ we use the *same* elementary row operations to reduce A to I_3. We can therefore save a significant amount of work by combining the foregoing operations as follows:

$$\left[\begin{array}{c|ccc} \mathbf{A} & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right] \sim \overset{\text{ERO}}{\cdots} \sim \begin{bmatrix} 1 & 0 & 0 & a_1 & b_1 & c_1 \\ 0 & 1 & 0 & a_2 & b_2 & c_2 \\ 0 & 0 & 1 & a_3 & b_3 & c_3 \end{bmatrix}.$$

The generalization to the $n \times n$ case is immediate. We form the $n \times 2n$ matrix $[A \quad I_n]$ and reduce A to I_n using elementary row operations. Schematically,

$$[A \quad I_n] \sim \overset{\text{ERO}}{\cdots} \sim [I_n \quad A^{-1}].$$

This method of finding A^{-1} is called the **Gauss-Jordan technique.**

Remark Notice that if we are given an $n \times n$ matrix A, we likely will not know from the outset whether $\text{rank}(A) = n$, hence we will not know whether A^{-1} exists. However, if at any stage in the row reduction of $[A \quad I_n]$ we find that $\text{rank}(A) < n$, then it will follow from Theorem 2.6.5 that A is not invertible.

Example 2.6.7 Find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix}$.

Solution: Using the Gauss-Jordan technique, we proceed as follows.

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 5 & -1 & 0 & 0 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -10 & -3 & 0 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -14 & -3 & -2 & 1 \end{bmatrix}$$

$$\overset{3}{\sim} \begin{bmatrix} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{14} & \frac{1}{7} & -\frac{1}{14} \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{14} & -\frac{8}{7} & \frac{1}{14} \\ 0 & 1 & 0 & -\frac{3}{7} & \frac{5}{7} & \frac{1}{7} \\ 0 & 0 & 1 & \frac{3}{14} & \frac{1}{7} & -\frac{1}{14} \end{bmatrix}.$$

Thus,

$$A^{-1} = \begin{bmatrix} \frac{11}{14} & -\frac{8}{7} & \frac{1}{14} \\ -\frac{3}{7} & \frac{5}{7} & \frac{1}{7} \\ \frac{3}{14} & \frac{1}{7} & -\frac{1}{14} \end{bmatrix}.$$

We leave it as an exercise to confirm that $AA^{-1} = A^{-1}A = I_3$.

1. $A_{13}(-3)$ **2**. $A_{21}(-1)$, $A_{23}(-2)$ **3**. $M_3(-1/14)$ **4**. $A_{31}(-1)$, $A_{32}(-2)$

□

Example 2.6.8 Continuing the previous example, use A^{-1} to solve the system

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2, \\ x_2 + 2x_3 &= 1, \\ 3x_1 + 5x_2 - x_3 &= 4. \end{aligned}$$

Solution: The system can be written as

$$A\mathbf{x} = \mathbf{b},$$

where A is the matrix in the previous example, and

$$\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Since A is invertible, the system has a unique solution that can be written as $\mathbf{x} = A^{-1}\mathbf{b}$. Thus, from the previous example we have

$$\mathbf{x} = \begin{bmatrix} \frac{11}{14} & -\frac{8}{7} & \frac{1}{14} \\ -\frac{3}{7} & \frac{5}{7} & \frac{1}{7} \\ \frac{3}{14} & \frac{1}{7} & -\frac{1}{14} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ \frac{3}{7} \\ \frac{2}{7} \end{bmatrix}.$$

Consequently, $x_1 = \frac{5}{7}$, $x_2 = \frac{3}{7}$, and $x_3 = \frac{2}{7}$, so that the solution to the system is $\left(\frac{5}{7}, \frac{3}{7}, \frac{2}{7}\right)$. □

We now return to more theoretical information pertaining to the inverse of a matrix.

Properties of the Inverse

The inverse of an $n \times n$ matrix satisfies the properties stated in the following theorem, which should be committed to memory:

Theorem 2.6.9 Let A and B be invertible $n \times n$ matrices. Then

1. A^{-1} is invertible and $(A^{-1})^{-1} = A$.
2. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof The proof of each result consists of verifying that the appropriate matrix products yield the identity matrix.

1. We must verify that

$$A^{-1}A = I_n \qquad \text{and} \qquad AA^{-1} = I_n.$$

Both of these follow directly from Definition 2.6.2.

2. We must verify that

$$(AB)(B^{-1}A^{-1}) = I_n \qquad \text{and} \qquad (B^{-1}A^{-1})(AB) = I_n.$$

We establish the first equality, leaving the second equation as an exercise. We have

$$(AB)(B^{-1})(A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

3. We must verify that

$$A^T(A^{-1})^T = I_n \qquad \text{and} \qquad (A^{-1})^T A^T = I_n.$$

Again, we prove the first part, leaving the second part as an exercise.
First recall from Theorem 2.2.21 that $A^T B^T = (BA)^T$. Using this property with $B = A^{-1}$ yields

$$A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n. \qquad \blacksquare$$

The proof of property 2 of Theorem 2.6.9 can easily be extended to a statement about invertibility of a product of an arbitrary finite number of matrices. More precisely, we have the following.

Corollary 2.6.10 Let $A_1, A_2, \ldots, A_k$ be invertible $n \times n$ matrices. Then $A_1A_2 \cdots A_k$ is invertible, and

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}.$$

Proof The proof is left as an exercise (Problem 28). $\blacksquare$

Some Further Theoretical Results

Finally, in this section, we establish two results that will be required in Section 2.7 and also in a proof that arises in Section 3.2.

Theorem 2.6.11 Let A and B be $n \times n$ matrices. If $AB = I_n$, then both A and B are invertible and $B = A^{-1}$.

Proof Let $\mathbf{b}$ be an arbitrary column n-vector. Then, since $AB = I_n$, we have

$$A(B\mathbf{b}) = I_n\mathbf{b} = \mathbf{b}.$$

Consequently, for *every* $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = B\mathbf{b}$. But this implies that $\text{rank}(A) = n$. To see why, suppose that $\text{rank}(A) < n$, and let A^* denote a row-echelon form of A. Note that the last row of A^* is zero. Choose $\mathbf{b}^*$ to be any column

n-vector whose last component is nonzero. Then, since $\text{rank}(A) < n$, it follows that the system

$$A^*\mathbf{x} = \mathbf{b}^*$$

is inconsistent. But, applying to the augmented matrix $[A^* \quad \mathbf{b}^*]$ the inverse row operations that reduced A to row-echelon form yields $[A \quad \mathbf{b}]$ for some $\mathbf{b}$. Since $A\mathbf{x} = \mathbf{b}$ has the same solution set as $A^*\mathbf{x} = \mathbf{b}^*$, it follows that $A\mathbf{x} = \mathbf{b}$ is inconsistent. We therefore have a contradiction, and so it must be the case that $\text{rank}(A) = n$, and therefore that A is invertible by Theorem 2.6.5.

We now establish that[8] $A^{-1} = B$. Since $AB = I_n$ by assumption, we have

$$A^{-1} = A^{-1}I_n = A^{-1}(AB) = (A^{-1}A)B = I_nB = B,$$

as required. It now follows directly from property 1 of Theorem 2.6.9 that B is invertible with inverse A. ■

Corollary 2.6.12 Let A and B be $n \times n$ matrices. If AB is invertible, then both A and B are invertible.

Proof If we let $C = B(AB)^{-1}$ and $D = AB$, then

$$AC = AB(AB)^{-1} = DD^{-1} = I_n.$$

It follows from Theorem 2.6.11 that A is invertible. Similarly, if we let $C = (AB)^{-1}A$, then

$$CB = (AB)^{-1}AB = I_n.$$

Once more we can apply Theorem 2.6.11 to conclude that B is invertible. ■

Exercises for 2.6

Key Terms

Inverse, Invertible, Singular, Nonsingular, Gauss-Jordan technique.

Skills

- Be able to check directly whether or not two matrices A and B are inverses of each other.
- Be able to find the inverse of an invertible matrix via the Gauss-Jordan technique.
- Be able to use the inverse of a coefficient matrix of a linear system in order to solve the system.
- Know the basic properties related to how the inverse operation behaves with respect to itself, multiplication, and transpose (Theorem 2.6.9).

True-False Review

For Questions 1–10, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. An invertible matrix is also known as a singular matrix.

[8]Note that it now makes sense to speak of A^{-1}, whereas prior to proving in the preceding paragraph that A is invertible, it would not have been legal to use the notation A^{-1}.

2. Every square matrix that does not contain a row of zeros is invertible.

3. A linear system $A\mathbf{x} = \mathbf{b}$ with an $n \times n$ invertible coefficient matrix A has a unique solution.

4. If A is a matrix such that there exists a matrix B with $AB = I_n$, then A is invertible.

5. If A and B are invertible $n \times n$ matrices, then so is $A + B$.

6. If A and B are invertible $n \times n$ matrices, then so is AB.

7. If A is an invertible matrix such that $A^2 = A$, then A is the identity matrix.

8. If A is an $n \times n$ invertible matrix and B and C are $n \times n$ matrices such that $AB = AC$, then $B = C$.

9. If A is a 5×5 matrix of rank 4, then A is not invertible.

10. If A is a 6×6 matrix of rank 6, then A is invertible.

Problems

For Problems 1–3 verify by direct multiplication that the given matrices are inverses of one another.

1. $A = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}$.

2. $A = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix}$.

3. $A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix}$.

For Problems 4–16, determine A^{-1}, if possible, using the Gauss-Jordan method. If A^{-1} exists, check your answer by verifying that $AA^{-1} = I_n$.

4. $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

5. $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$.

6. $A = \begin{bmatrix} 1 & -i \\ -1+i & 2 \end{bmatrix}$.

7. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

8. $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 11 \\ 4 & -3 & 10 \end{bmatrix}$.

9. $A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}$.

10. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

11. $A = \begin{bmatrix} 4 & 2 & -13 \\ 2 & 1 & -7 \\ 3 & 2 & 4 \end{bmatrix}$.

12. $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & -2 \\ -1 & 1 & 4 \end{bmatrix}$.

13. $A = \begin{bmatrix} 1 & i & 2 \\ 1+i & -1 & 2i \\ 2 & 2i & 5 \end{bmatrix}$.

14. $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

15. $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & -4 \\ 3 & -1 & 7 & 8 \\ 1 & 0 & 3 & 5 \end{bmatrix}$.

16. $A = \begin{bmatrix} 0 & -2 & -1 & -3 \\ 2 & 0 & 2 & 1 \\ 1 & -2 & 0 & 2 \\ 3 & -1 & -2 & 0 \end{bmatrix}$.

17. Let

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}.$$

Find the second column vector of A^{-1} without determining the whole inverse.

For Problems 18–22, use A^{-1} to find the solution to the given system.

18.
$$\begin{aligned} x_1 + 3x_2 &= 1, \\ 2x_1 + 5x_2 &= 3. \end{aligned}$$

19.
$$\begin{aligned} x_1 + x_2 - 2x_3 &= -2, \\ x_2 + x_3 &= 3, \\ 2x_1 + 4x_2 - 3x_3 &= 1. \end{aligned}$$

20.
$$\begin{aligned} x_1 - 2ix_2 &= 2, \\ (2-i)x_1 + 4ix_2 &= -i. \end{aligned}$$

21. $$\begin{aligned} 3x_1 + 4x_2 + 5x_3 &= 1, \\ 2x_1 + 10x_2 + x_3 &= 1, \\ 4x_1 + x_2 + 8x_3 &= 1. \end{aligned}$$

22. $$\begin{aligned} x_1 + x_2 + 2x_3 &= 12, \\ x_1 + 2x_2 - x_3 &= 24, \\ 2x_1 - x_2 + x_3 &= -36. \end{aligned}$$

An $n \times n$ matrix A is called **orthogonal** if $A^T = A^{-1}$. For Problems 23–26, show that the given matrices are orthogonal.

23. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

24. $A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$.

25. $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$.

26. $A = \dfrac{1}{1+2x^2}\begin{bmatrix} 1 & -2x & 2x^2 \\ 2x & 1-2x^2 & -2x \\ 2x^2 & 2x & 1 \end{bmatrix}$.

27. Complete the proof of Theorem 2.6.9 by verifying the remaining properties in parts 2 and 3.

28. Prove Corollary 2.6.10.

For Problems 29–30, use properties of the inverse to prove the given statement.

29. If A is an $n \times n$ invertible *symmetric* matrix, then A^{-1} is symmetric.

30. If A is an $n \times n$ invertible *skew-symmetric* matrix, then A^{-1} is skew-symmetric.

31. Let A be an $n \times n$ matrix with $A^4 = 0$. Prove that $I_n - A$ is invertible with

$$(I_n - A)^{-1} = I_n + A + A^2 + A^3.$$

32. Prove that if A, B, C are $n \times n$ matrices satisfying $BA = I_n$ and $AC = I_n$, then $B = C$.

33. If A, B, C are $n \times n$ matrices satisfying $BA = I_n$ and $CA = I_n$, does it follow that $B = C$? Justify your answer.

34. Consider the general 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and let $\Delta = a_{11}a_{22} - a_{12}a_{21}$ with $a_{11} \neq 0$. Show that if $\Delta \neq 0$,

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

The quantity Δ defined above is referred to as the determinant of A. We will investigate determinants in more detail in the next chapter.

35. Let A be an $n \times n$ matrix, and suppose that we have to solve the p linear systems

$$A\mathbf{x}_i = \mathbf{b}_i, \qquad i = 1, 2, \dots, p$$

where the $\mathbf{b}_i$ are given. Devise an efficient method for solving these systems.

36. Use your method from the previous problem to solve the three linear systems

$$A\mathbf{x}_i = \mathbf{b}_i, \qquad i = 1, 2, 3$$

if

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 1 & 1 & 6 \end{bmatrix}, \qquad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

$$\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \qquad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}.$$

37. Let A be an $m \times n$ matrix with $m \leq n$.

(a) If $\text{rank}(A) = m$, prove that there exists a matrix B satisfying $AB = I_m$. Such a matrix is called a **right inverse** of A.

(b) If

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \end{bmatrix},$$

determine all right inverses of A.

◊ For Problems 38–39, reduce the matrix $[A \quad I_n]$ to reduced row-echelon form and thereby determine, if possible, the inverse of A.

38. $A = \begin{bmatrix} 5 & 9 & 17 \\ 7 & 21 & 13 \\ 27 & 16 & 8 \end{bmatrix}$.

39. A is a randomly generated 4×4 matrix.

◊ For Problems 40–42, use built-in functions of some form of technology to determine $\text{rank}(A)$ and, if possible, A^{-1}.

40. $A = \begin{bmatrix} 3 & 5 & -7 \\ 2 & 5 & 9 \\ 13 & -11 & 22 \end{bmatrix}$.

41. $A = \begin{bmatrix} 7 & 13 & 15 & 21 \\ 9 & -2 & 14 & 23 \\ 17 & -27 & 22 & 31 \\ 19 & -42 & 21 & 33 \end{bmatrix}.$

42. A is a randomly generated 5×5 matrix.

43. ◇ For the system in Problem 21, determine A^{-1} and use it to solve the system.

44. ◇ Consider the $n \times n$ **Hilbert** matrix

$$H_n = \left[\frac{1}{i+j-1}\right], \qquad 1 \leq i, \quad j \leq n.$$

(a) Determine H_4 and show that it is invertible.

(b) Find H_4^{-1} and use it to solve $H_4\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = [2, -1, 3, 5]^T$.

2.7 Elementary Matrices and the LU Factorization

We now introduce some matrices that can be used to perform elementary row operations on a matrix. Although they are of limited computational use, they do play a significant role in linear algebra and its applications.

DEFINITION 2.7.1

Any matrix obtained by performing a single elementary row operation on the identity matrix is called an **elementary matrix**.

In particular, an elementary matrix is always a square matrix. In general we will denote elementary matrices by E. If we are describing a specific elementary matrix, then in keeping with the notation introduced previously for elementary row operations, we will use the following notation for the three types of elementary matrices:

Type 1: P_{ij}—permute rows i and j in I_n.

Type 2: $\mathrm{M}_i(k)$—multiply row i of I_n by the nonzero scalar k.

Type 3: $\mathrm{A}_{ij}(k)$—add k times row i of I_n to row j of I_n.

Example 2.7.2 Write all 2×2 elementary matrices.

Solution: From Definition 2.7.1 and using the notation introduced above, we have

1. Permutation matrix: $\mathrm{P}_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

2. Scaling matrices: $\mathrm{M}_1(k) = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2(k) = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}.$

3. Row combinations: $\mathrm{A}_{12}(k) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \quad A_{21}(k) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$

□

We leave it as an exercise to verify that the $n \times n$ elementary matrices have the following structure: