41. $A = \begin{bmatrix} 7 & 13 & 15 & 21 \\ 9 & -2 & 14 & 23 \\ 17 & -27 & 22 & 31 \\ 19 & -42 & 21 & 33 \end{bmatrix}$.

42. A is a randomly generated 5×5 matrix.

43. ◇ For the system in Problem 21, determine A^{-1} and use it to solve the system.

44. ◇ Consider the $n \times n$ **Hilbert** matrix

$$H_n = \left[\frac{1}{i+j-1}\right], \qquad 1 \leq i, \quad j \leq n.$$

(a) Determine H_4 and show that it is invertible.

(b) Find H_4^{-1} and use it to solve $H_4\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = [2, -1, 3, 5]^T$.

2.7 Elementary Matrices and the LU Factorization

We now introduce some matrices that can be used to perform elementary row operations on a matrix. Although they are of limited computational use, they do play a significant role in linear algebra and its applications.

DEFINITION 2.7.1

Any matrix obtained by performing a single elementary row operation on the identity matrix is called an **elementary matrix**.

In particular, an elementary matrix is always a square matrix. In general we will denote elementary matrices by E. If we are describing a specific elementary matrix, then in keeping with the notation introduced previously for elementary row operations, we will use the following notation for the three types of elementary matrices:

Type 1: P_{ij}—permute rows i and j in I_n.

Type 2: $M_i(k)$—multiply row i of I_n by the nonzero scalar k.

Type 3: $A_{ij}(k)$—add k times row i of I_n to row j of I_n.

Example 2.7.2 Write all 2×2 elementary matrices.

Solution: From Definition 2.7.1 and using the notation introduced above, we have

1. Permutation matrix: $P_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

2. Scaling matrices: $M_1(k) = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $M_2(k) = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$.

3. Row combinations: $A_{12}(k) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, $A_{21}(k) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.

□

We leave it as an exercise to verify that the $n \times n$ elementary matrices have the following structure:

P_{ij}: ones along main diagonal except (i, i) and (j, j), ones in the (i, j) and (j, i) positions, and zeros elsewhere.

$M_i(k)$: the diagonal matrix $\text{diag}(1, 1, \ldots, k, \ldots, 1)$, where k appears in the (i, i) position.

$A_{ij}(k)$: ones along the main diagonal, k in the (j, i) position, and zeros elsewhere.

A key point to note about elementary matrices is the following:

> Premultiplying an $n \times p$ matrix A by an $n \times n$ elementary matrix E has the effect of performing the corresponding elementary row operation on A.

Rather than proving this statement, which we leave as an exercise, we illustrate with an example.

Example 2.7.3 If $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 7 & 5 \end{bmatrix}$, then, for example,

$$M_1(k)A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 4 \\ 2 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 3k & -k & 4k \\ 2 & 7 & 5 \end{bmatrix}.$$

Similarly,

$$A_{21}(k)A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 4 \\ 2 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 3+2k & -1+7k & 4+5k \\ 2 & 7 & 5 \end{bmatrix}.$$

□

Since elementary row operations can be performed on a matrix by premultiplication by an appropriate elementary matrix, it follows that any matrix A can be reduced to row-echelon form by multiplication by a sequence of elementary matrices. Schematically we can therefore write

$$E_k E_{k-1} \cdots E_2 E_1 A = U,$$

where U denotes a row-echelon form of A and the E_i are elementary matrices.

Example 2.7.4 Determine elementary matrices that reduce $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ to row-echelon form.

Solution: We can reduce A to row-echelon form using the following sequence of elementary row operations:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & -5 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-2)$ **3.** $M_2(-\frac{1}{5})$

Consequently,

$$M_2(-\tfrac{1}{5})A_{12}(-2)P_{12}A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix},$$

which we can verify by direct multiplication:

$$\begin{aligned} M_2(-\tfrac{1}{5})A_{12}(-2)P_{12}A &= \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

□

Since any elementary row operation is reversible, it follows that each elementary matrix is invertible. Indeed, in the 2×2 case it is easy to see that

$$P_{12}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad M_1(k)^{-1} = \begin{bmatrix} 1/k & 0 \\ 0 & 1 \end{bmatrix}, \qquad M_2(k)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/k \end{bmatrix},$$
$$A_{12}(k)^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}, \qquad A_{21}(k)^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}.$$

We leave it as an exercise to verify that in the $n \times n$ case, we have:

$$\boxed{M_i(k)^{-1} = M_i(1/k), \qquad P_{ij}^{-1} = P_{ij}, \qquad A_{ij}(k)^{-1} = A_{ij}(-k)}$$

Now consider an *invertible* $n \times n$ matrix A. Since the unique reduced row-echelon form of such a matrix is the identity matrix I_n, it follows from the preceding discussion that there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = I_n.$$

But this implies that

$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1,$$

and hence,

$$A = (A^{-1})^{-1} = (E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

which is a product of elementary matrices. So any invertible matrix is a product of elementary matrices. Conversely, since elementary matrices are invertible, a product of elementary matrices is a product of invertible matrices, hence is invertible by Corollary 2.6.10. Therefore, we have established the following.

Theorem 2.7.5 Let A be an $n \times n$ matrix. Then A is invertible if and only if A is a product of elementary matrices.

The LU Decomposition of an Invertible Matrix [9]

For the remainder of this section, we restrict our attention to invertible $n \times n$ matrices. In reducing such a matrix to row-echelon form, we have always placed leading ones on the main diagonal in order that we obtain a row-echelon matrix. We now lift the requirement that the main diagonal of the row-echelon form contain ones. As a consequence, the matrix that results from row reduction will be an upper triangular matrix but will not necessarily be in row-echelon form. Furthermore, reduction to such an upper triangular form can be accomplished without the use of Type 2 row operations.

[9] The material in the remainder of this section is not used elsewhere in the text.

Example 2.7.6 Use elementary row operations to reduce the matrix

$$A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$$

to upper triangular form.

Solution: The given matrix can be reduced to upper triangular form using the following sequence of elementary row operations:

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & 5 & 2 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & \frac{9}{2} & \frac{5}{2} \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 2 & 5 & 3 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & 0 & -2 \end{bmatrix}.$$

1. $A_{12}(-\frac{3}{2})$, $A_{13}(\frac{1}{2})$ **2.** $A_{23}(\frac{9}{13})$ □

When using elementary row operations of Type 3, the multiple of a specific row that is *subtracted* from row i to put a zero in the (i, j) position is called a **multiplier** and denoted m_{ij}. Thus, in the preceding example, there are three multipliers—namely,

$$m_{21} = \tfrac{3}{2}, \qquad m_{31} = -\tfrac{1}{2}, \qquad m_{32} = -\tfrac{9}{13}.$$

The multipliers will be used in the forthcoming discussion.

In Example 2.7.6 we were able to reduce A to upper triangular form using only row operations of Type 3. This is not always the case. For example, the matrix

$$\begin{bmatrix} 0 & 5 \\ 3 & 2 \end{bmatrix}$$

requires that the two rows be permuted to obtain an upper triangular form. *For the moment, however, we will restrict our attention to invertible matrices A for which the reduction to upper triangular form can be accomplished without permuting rows.* In this case, we can therefore reduce A to upper triangular form using row operations of Type 3 only. Furthermore, throughout the reduction process, we can restrict ourselves to Type 3 operations that add multiples of a row to rows *beneath* that row, by simply performing the row operations column by column, from left to right. According to our description of the elementary matrices $A_{ij}(k)$, our reduction process therefore uses only elementary matrices that are *unit lower triangular*. More specifically, in terms of elementary matrices we have

$$E_k E_{k-1} \cdots E_2 E_1 A = U,$$

where $E_k, E_{k-1}, \ldots, E_2, E_1$ are unit lower triangular Type 3 elementary matrices and U is an upper triangular matrix. Since each elementary matrix is invertible, we can write the preceding equation as

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U. \tag{2.7.1}$$

But, as we have already argued, each of the elementary matrices in (2.7.1) is a unit lower triangular matrix, and we know from Corollary 2.2.23 that the product of two unit lower triangular matrices is also a unit lower triangular matrix. Consequently, (2.7.1) can be written as

$$A = LU, \tag{2.7.2}$$

where

$$L = E_1^{-1}E_2^{-1}\cdots E_k^{-1} \tag{2.7.3}$$

is a unit lower triangular matrix and U is an upper triangular matrix. Equation (2.7.2) is referred to as the **LU factorization of A.** It can be shown (Problem 29) that this LU factorization is unique.

Example 2.7.7 Determine the LU factorization of the matrix

$$A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}.$$

Solution: Using the results of Example 2.7.6, we can write

$$E_3E_2E_1A = \begin{bmatrix} 2 & 5 & 3 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & 0 & -2 \end{bmatrix},$$

where

$$E_1 = \mathrm{A}_{12}(-\tfrac{3}{2}), \qquad E_2 = \mathrm{A}_{13}(\tfrac{1}{2}), \qquad \text{and} \qquad E_3 = \mathrm{A}_{23}(\tfrac{9}{13}).$$

Therefore,

$$U = \begin{bmatrix} 2 & 5 & 3 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & 0 & -2 \end{bmatrix}$$

and from (2.7.3),

$$L = E_1^{-1}E_2^{-1}\cdots E_k^{-1}. \tag{2.7.4}$$

Computing the inverses of the elementary matrices, we have

$$E_1^{-1} = \mathrm{A}_{12}(\tfrac{3}{2}), \qquad E_2^{-1} = \mathrm{A}_{13}(-\tfrac{1}{2}), \qquad \text{and} \qquad E_3^{-1} = \mathrm{A}_{23}(-\tfrac{9}{13}).$$

Substituting these results into (2.7.4) yields

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{9}{13} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{9}{13} & 1 \end{bmatrix}.$$

Consequently,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{9}{13} & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 3 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & 0 & -2 \end{bmatrix}$$

which is easily verified by a matrix multiplication. □

Computing the lower triangular matrix L in the LU factorization of A using (2.7.3) can require a significant amount of work. However, if we look carefully at the matrix L in Example 2.7.7, we see that the elements beneath the leading diagonal are just the corresponding multipliers. That is, if l_{ij} denotes the (i, j) element of the matrix L, then

$$l_{ij} = m_{ij}, \qquad i > j. \tag{2.7.5}$$

Furthermore, it can be shown that this relationship holds in general. Consequently, we do not need to use (2.7.3) to obtain L. Instead we use row operations of Type 3 to reduce A to upper triangular form, and then we can use (2.7.5) to obtain L directly.

Example 2.7.8 Determine the LU decomposition for the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 & 2 \\ 5 & -1 & 2 & 1 \\ 3 & 2 & 6 & -5 \\ -1 & 1 & 3 & 2 \end{bmatrix}.$$

Solution: To determine U, we reduce A to upper triangular form using only row operations of Type 3 in which we add multiples of a given row only to rows *below* the given row.

$$A \overset{1}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & \frac{13}{2} & -\frac{1}{2} & -4 \\ 0 & \frac{13}{2} & \frac{9}{2} & -8 \\ 0 & -\frac{1}{2} & \frac{7}{2} & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & \frac{13}{2} & -\frac{1}{2} & -4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & \frac{45}{13} & \frac{35}{13} \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & \frac{13}{2} & -\frac{1}{2} & -4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & \frac{71}{13} \end{bmatrix} = U.$$

Row Operations	**Corresponding Multipliers**
(1) $A_{12}(-\frac{5}{2})$, $A_{13}(-\frac{3}{2})$, $A_{14}(\frac{1}{2})$	$m_{21} = \frac{5}{2}$, $m_{31} = \frac{3}{2}$, $m_{41} = -\frac{1}{2}$
(2) $A_{23}(-1)$, $A_{24}(\frac{1}{13})$	$m_{32} = 1$, $m_{42} = -\frac{1}{13}$
(3) $A_{34}(-\frac{9}{13})$	$m_{43} = \frac{9}{13}$

Consequently, from (2.7.4),

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{5}{2} & 1 & 0 & 0 \\ \frac{3}{2} & 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{13} & \frac{9}{13} & 1 \end{bmatrix}.$$

We leave it as an exercise to verify that $LU = A$. □

The question undoubtedly in the reader's mind is: What is the use of the LU decomposition? In order to answer this question, consider the $n \times n$ system of linear equation $A\mathbf{x} = \mathbf{b}$, where $A = LU$. If we write the system as

$$LU\mathbf{x} = \mathbf{b}$$

and let $U\mathbf{x} = \mathbf{y}$, then solving $A\mathbf{x} = \mathbf{b}$ is equivalent to solving the pair of equations

$$L\mathbf{y} = \mathbf{b},$$
$$U\mathbf{x} = \mathbf{y}.$$

Owing to the triangular form of each of the coefficient matrices L and U, these systems can be solved easily—the first by "forward" substitution and the second by back substitution. In the case when we have a single right-hand-side vector $\mathbf{b}$, the LU factorization for solving the system has no advantage over Gaussian elimination. However, if we require the solution of several systems of equations with the same coefficient matrix A, say

$$A\mathbf{x}_i = \mathbf{b}_i, \qquad i = 1, 2, \dots, p$$

then it is more efficient to compute the LU factorization of A once, and then successively solve the triangular systems

$$\left.\begin{aligned} L\mathbf{y}_i &= \mathbf{b}_i, \\ U\mathbf{x}_i &= \mathbf{y}_i. \end{aligned}\right\} \qquad i = 1, 2, \dots, p.$$

Example 2.7.9 Use the LU decomposition of

$$A = \begin{bmatrix} 2 & -3 & 1 & 2 \\ 5 & -1 & 2 & 1 \\ 3 & 2 & 6 & -5 \\ -1 & 1 & 3 & 2 \end{bmatrix}$$

to solve the system $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 5 \\ 7 \end{bmatrix}$.

Solution: We have shown in the previous example that $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{5}{2} & 1 & 0 & 0 \\ \frac{3}{2} & 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{13} & \frac{9}{13} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & \frac{13}{2} & -\frac{1}{2} & -4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & \frac{71}{13} \end{bmatrix}.$$

We now solve the two triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. Using forward substitution on the first of these systems, we have

$$\begin{aligned} y_1 &= 2, \qquad y_2 = -3 - \tfrac{5}{2}y_1 = -8, \\ y_3 &= 5 - \tfrac{3}{2}y_1 - y_2 = 5 - 3 + 8 = 10, \\ y_4 &= 7 + \tfrac{1}{2}y_1 + \tfrac{1}{13}y_2 - \tfrac{9}{13}y_3 = 8 - \tfrac{8}{13} - \tfrac{90}{13} = \tfrac{6}{13}. \end{aligned}$$

Solving $U\mathbf{x} = \mathbf{y}$ via back substitution yields

$$x_4 = \frac{13}{71}y_4 = \frac{6}{71}, \qquad x_3 = \frac{1}{5}(y_3 + 4x_4) = \frac{1}{5}\left(10 + \frac{24}{71}\right) = \frac{734}{355},$$

$$x_2 = \frac{2}{13}\left(y_2 + \frac{1}{2}x_3 + 4x_4\right) = \frac{2}{13}\left(-8 + \frac{367}{355} + \frac{24}{71}\right) = -\frac{362}{355},$$

$$x_1 = \frac{1}{2}\left(y_1 + 3x_2 - x_3 - 2x_4\right) = \frac{1}{2}\left(2 - \frac{1086}{355} - \frac{734}{355} - \frac{12}{71}\right) = -\frac{117}{71}.$$

Consequently,

$$\mathbf{x} = \left(-\frac{117}{71}, -\frac{362}{355}, \frac{734}{355}, \frac{6}{71}\right).$$

□

In the more general case when row interchanges are required to reduce an invertible matrix A to upper triangular form, it can be shown that A has a factorization of the form

$$A = PLU, \tag{2.7.6}$$

where P is an appropriate product of elementary permutation matrices, L is a unit lower triangular matrix, and U is an upper triangular matrix. From the properties of the elementary permutation matrices, it follows (see Problem 27), that $P^{-1} = P^T$. Using (2.7.6) the linear system $A\mathbf{x} = \mathbf{b}$ can be written as

$$PLU\mathbf{x} = \mathbf{b},$$

or equivalently,

$$LU\mathbf{x} = P^T\mathbf{b}.$$

Consequently, to solve $A\mathbf{x} = \mathbf{b}$ in this case we can solve the two triangular systems

$$\begin{cases} L\mathbf{y} = P^T\mathbf{b}, \\ U\mathbf{x} = \mathbf{y}. \end{cases}$$

For a full discussion of this and other factorizations of $n \times n$ matrices, and their applications, the reader is referred to more advanced texts on linear algebra or numerical analysis [for example, B. Noble and J. W.Daniel, *Applied Linear Algebra* (Englewood Cliffs, N.J., Prentice Hall, 1988); J. Ll. Morris, *Computational Methods in Elementary Numerical Analysis* (New York: Wiley, 1983)].

Exercises for 2.7

Key Terms

Elementary matrix, Multiplier, LU Factorization of a matrix.

Skills

- Be able to determine whether or not a given matrix is an elementary matrix.
- Know the form for the permutation matrices, scaling matrices, and row combination matrices.
- Be able to write down the inverse of an elementary matrix without any computation.
- Be able to determine elementary matrices that reduce a given matrix to row-echelon form.
- Be able to express an invertible matrix as a product of elementary matrices.
- Be able to determine the multipliers of a matrix.
- Be able to determine the LU factorization of a matrix.
- Be able to use the LU factorization of a matrix A to solve a linear system $A\mathbf{x} = \mathbf{b}$.

True-False Review

For Questions 1–10, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. Every elementary matrix is invertible.

2. A product of elementary matrices is an elementary matrix.

3. Every matrix can be expressed as a product of elementary matrices.

4. If A is an $m \times n$ matrix and E is an $m \times m$ elementary matrix, then the matrices A and EA have the same rank.

5. If P_{ij} is a permutation matrix, then $P_{ij}^2 = P_{ij}$.

6. If E_1 and E_2 are $n \times n$ elementary matrices, then $E_1E_2 = E_2E_1$.

7. If E_1 and E_2 are $n \times n$ elementary matrices *of the same type*, then $E_1E_2 = E_2E_1$.

8. Every matrix has an LU factorization.

9. In the LU factorization of a matrix A, the matrix L is a unit lower triangular matrix and the matrix U is a unit upper triangular matrix.

10. A 4×4 matrix A that has an LU factorization has 10 multipliers.

Problems

1. Write all 3×3 elementary matrices and their inverses.

For Problems 2–5, determine elementary matrices that reduce the given matrix to row-echelon form.

2. $\begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix}$.

3. $\begin{bmatrix} 5 & 8 & 2 \\ 1 & 3 & -1 \end{bmatrix}$.

4. $\begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$.

For Problems 6–12, express the matrix A as a product of elementary matrices.

6. $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

7. $A = \begin{bmatrix} -2 & -3 \\ 5 & 7 \end{bmatrix}$.

8. $A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$.

9. $A = \begin{bmatrix} 4 & -5 \\ 1 & 4 \end{bmatrix}$.

10. $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$.

11. $A = \begin{bmatrix} 0 & -4 & -2 \\ 1 & -1 & 3 \\ -2 & 2 & 2 \end{bmatrix}$.

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 8 & 0 \\ 3 & 4 & 5 \end{bmatrix}$.

13. Determine elementary matrices $E_1, E_2, \ldots, E_k$ that reduce

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

to reduced row-echelon form. Verify by direct multiplication that $E_1E_2 \cdots E_kA = I_2$.

14. Determine a Type 3 lower triangular elementary matrix E_1 that reduces

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix}$$

to upper triangular form. Use Equation (2.7.3) to determine L and verify Equation (2.7.2).

For Problems 15–20, determine the LU factorization of the given matrix. Verify your answer by computing the product LU.

15. $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$.

16. $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

17. $A = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix}$.

18. $A = \begin{bmatrix} 5 & 2 & 1 \\ -10 & -2 & 3 \\ 15 & 2 & -3 \end{bmatrix}$.

19. $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & -4 \\ 3 & -1 & 7 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix}$.

20. $A = \begin{bmatrix} 2 & -3 & 1 & 2 \\ 4 & -1 & 1 & 1 \\ -8 & 2 & 2 & -5 \\ 6 & 1 & 5 & 2 \end{bmatrix}$.

For Problems 21–24, use the LU factorization of A to solve the system $A\mathbf{x} = \mathbf{b}$.

21. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

22. $A = \begin{bmatrix} 1 & -3 & 5 \\ 3 & 2 & 2 \\ 2 & 5 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$.

23. $A = \begin{bmatrix} 2 & 2 & 1 \\ 6 & 3 & -1 \\ -4 & 2 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

24. $A = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 8 & 1 & 2 & 0 \\ 0 & 5 & 3 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 5 \end{bmatrix}$.

25. Use the LU factorization of

$$A = \begin{bmatrix} 2 & -1 \\ -8 & 3 \end{bmatrix}$$

to solve each of the systems $A\mathbf{x}_i = \mathbf{b}_i$ if

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 5 \\ -9 \end{bmatrix}.$$

26. Use the LU factorization of

$$A = \begin{bmatrix} -1 & 4 & 2 \\ 3 & 1 & 4 \\ 5 & -7 & 1 \end{bmatrix}$$

to solve each of the systems $A\mathbf{x}_i = \mathbf{e}_i$ and thereby determine A^{-1}.

27. If $P = P_1 P_2 \cdots P_k$, where each P_i is an elementary permutation matrix, show that $P^{-1} = P^T$.

28. Prove that

(a) The inverse of an invertible upper triangular matrix is upper triangular. Repeat for an invertible lower triangular matrix.

(b) The inverse of a unit upper triangular matrix is unit upper triangular. Repeat for a unit lower triangular matrix.

29. In this problem, we prove that the LU decomposition of an invertible $n \times n$ matrix is unique in the sense that, if $A = L_1U_1$ and $A = L_2U_2$, where L_1, L_2 are unit lower triangular matrices and U_1, U_2 are upper triangular matrices, then $L_1 = L_2$ and $U_1 = U_2$.

(a) Apply Corollary 2.6.12 to conclude that L_2 and U_1 are invertible, and then use the fact that $L_1U_1 = L_2U_2$ to establish that $L_2^{-1}L_1 = U_2U_1^{-1}$.

(b) Use the result from (a) together with Theorem 2.2.22 and Corollary 2.2.23 to prove that $L_2^{-1}L_1 = I_n$ and $U_2U_1^{-1} = I_n$, from which the required result follows.

30. QR Factorization: It can be shown that any invertible $n \times n$ matrix has a factorization of the form

$$A = QR,$$

where Q and R are invertible, R is upper triangular, and Q satisfies $Q^TQ = I_n$ (i.e., Q is **orthogonal**). Determine an algorithm for solving the linear system $A\mathbf{x} = \mathbf{b}$ using this QR factorization.

◊ For Problems 31–33, use some form of technology to determine the LU factorization of the given matrix. Verify the factorization by computing the product LU.

31. $A = \begin{bmatrix} 3 & 5 & -2 \\ 2 & 7 & 9 \\ -5 & 5 & 11 \end{bmatrix}$.

32. $A = \begin{bmatrix} 27 & -19 & 32 \\ 15 & -16 & 9 \\ 23 & -13 & 51 \end{bmatrix}$.

33. $A = \begin{bmatrix} 34 & 13 & 19 & 22 \\ 53 & 17 & -71 & 20 \\ 21 & 37 & 63 & 59 \\ 81 & 93 & -47 & 39 \end{bmatrix}$.

2.8 The Invertible Matrix Theorem I

In Section 2.6, we defined an $n \times n$ invertible matrix A to be a matrix such that there exists an $n \times n$ matrix B satisfying $AB = BA = I_n$. There are, however, many other important and useful viewpoints on invertibility of matrices. Some of these we have already encountered in the preceding two sections, while others await us in later chapters. It is worthwhile to begin collecting a list of conditions on an $n \times n$ matrix A that are