38. $A = \begin{bmatrix} e^t & te^t & e^{-2t} \\ e^t & 2te^t & e^{-2t} \\ e^t & te^t & 2e^{-2t} \end{bmatrix}$.

39. If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix},$$

compute the matrix product $A \cdot \text{adj}(A)$. What can you conclude about $\det(A)$?

For Problems 40–43, use Cramer's rule to solve the given linear system.

40.
$$\begin{aligned} 2x_1 - 3x_2 &= 2, \\ x_1 + 2x_2 &= 4. \end{aligned}$$

41.
$$\begin{aligned} 3x_1 - 2x_2 + x_3 &= 4, \\ x_1 + x_2 - x_3 &= 2, \\ x_1 \qquad\quad + x_3 &= 1. \end{aligned}$$

42.
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0, \\ x_1 + 4x_2 - x_3 &= 0, \\ 2x_1 + x_2 - 3x_3 &= 0. \end{aligned}$$

43.
$$\begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 1, \\ 2x_1 \qquad\quad + x_3 \qquad &= 2, \\ x_1 + x_2 \qquad\quad - x_4 &= 0, \\ x_2 - 2x_3 + x_4 &= 3. \end{aligned}$$

44. Use Cramer's rule to determine x_1 and x_2 if

$$\begin{aligned} e^t x_1 + e^{-2t} x_2 &= 3\sin t, \\ e^t x_1 - 2e^{-2t} x_2 &= 4\cos t. \end{aligned}$$

45. Determine the value of x_2 such that

$$\begin{aligned} x_1 + 4x_2 - 2x_3 + x_4 &= 2, \\ 2x_1 + 9x_2 - 3x_3 - 2x_4 &= 5, \\ x_1 + 5x_2 + x_3 - x_4 &= 3, \\ 3x_1 + 14x_2 + 7x_3 - 2x_4 &= 6. \end{aligned}$$

46. Find all solutions to the system

$$\begin{aligned} (b+c)x_1 + a(x_2+x_3) &= a, \\ (c+a)x_1 + b(x_3+x_1) &= b, \\ (a+b)x_1 + c(x_1+x_2) &= c, \end{aligned}$$

where a, b, c are constants. Make sure you consider all cases (that is, those when there is a unique solution, an infinite number of solutions, and no solutions).

47. Prove Equation (3.3.3).

48. ◇ Let A be a randomly generated invertible 4×4 matrix. Verify the Cofactor Expansion Theorem for expansion along row 1.

49. ◇ Let A be a randomly generated 4×4 matrix. Verify Equation (3.3.3) when $i = 2$ and $j = 4$.

50. ◇ Let A be a randomly generated 5×5 matrix. Determine $\text{adj}(A)$ and compute $A \cdot \text{adj}(A)$. Use your result to determine $\det(A)$.

51. ◇ Solve the system of equations

$$\begin{aligned} 1.21x_1 + 3.42x_2 + 2.15x_3 &= 3.25, \\ 5.41x_1 + 2.32x_2 + 7.15x_3 &= 4.61, \\ 21.63x_1 + 3.51x_2 + 9.22x_3 &= 9.93. \end{aligned}$$

Round answers to two decimal places.

52. ◇ Use Cramer's rule to solve the system $A\mathbf{x} = \mathbf{b}$ if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 68 \\ -72 \\ -87 \\ 79 \\ 43 \end{bmatrix}.$$

53. Verify that $BA = I_n$ in the proof of Theorem 3.3.16.

3.4 Summary of Determinants

The primary aim of this section is to serve as a stand-alone introduction to determinants for readers who desire only a cursory review of the major facts pertaining to determinants. It may also be used as a review of the results derived in Sections 3.1–3.3.

Formulas for the Determinant

The determinant of an $n \times n$ matrix A, denoted $\det(A)$, is a scalar whose value can be obtained in the following manner.

1. If $A = [a_{11}]$, then $\det(A) = a_{11}$.

2. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

3. For $n > 2$, the determinant of A can be computed using either of the following formulas:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}, \tag{3.4.1}$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}, \tag{3.4.2}$$

where $C_{ij} = (-1)^{i+j}M_{ij}$, and M_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The formulas (3.4.1) and (3.4.2) are referred to as cofactor expansion along the ith row and cofactor expansion along the jth column, respectively. The determinants M_{ij} and C_{ij} are called the **minors** and **cofactors** of A, respectively. We also denote $\det(A)$ by

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

As an example, consider the general 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Using cofactor expansion along row 1, we have

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}. \tag{3.4.3}$$

We next compute the required cofactors:

$$C_{11} = +M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32},$$

$$C_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31}),$$

$$C_{13} = +M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}.$$

Inserting these expressions for the cofactors into Equation (3.4.3) yields

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}),$$

which can be written as

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

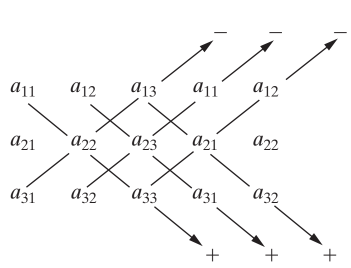

Figure 3.4.1: A schematic for obtaining the determinant of a 3×3 matrix $A = [a_{ij}]$.

Although we chose to use cofactor expansion along the first row to obtain the preceding formula, according to (3.4.1) and (3.4.2), the same result would have been obtained if we had chosen to expand along any row or column of A. A simple schematic for obtaining the terms in the determinant of a 3×3 matrix is given in Figure 3.4.1. By taking the product of the elements joined by each arrow and attaching the indicated sign to the result, we obtain the six terms in the determinant of the 3×3 matrix $A = [a_{ij}]$. Note that this technique for obtaining the terms in a 3×3 determinant *does not* generalize to determinants of larger matrices.

Example 3.4.1 Evaluate

$$\begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & 2 \\ 7 & 5 & 8 \end{vmatrix}.$$

Solution: In this case, the schematic given in Figure 3.4.1 is

$$\begin{matrix} 2 & -1 & 1 & 2 & -1 \\ 3 & 4 & 2 & 3 & 4 \\ 7 & 5 & 8 & 7 & 5 \end{matrix}$$

so that

$$\begin{aligned} \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & 2 \\ 7 & 5 & 8 \end{vmatrix} &= (2)(4)(8) + (-1)(2)(7) + (1)(3)(5) - (7)(4)(1) - (5)(2)(2) - (8)(3)(-1) \\ &= 41. \end{aligned}$$

□

Properties of Determinants

Let A and B be $n \times n$ matrices. The determinant has the following properties:

P1. If B is obtained by permuting two rows (or columns) of A, then

$$\det(B) = -\det(A).$$

P2. If B is obtained by multiplying any row (or column) of A by a scalar k, then

$$\det(B) = k \det(A).$$

P3. If B is obtained by adding a multiple of any row (or column) of A to another row (or column) of A, then

$$\det(B) = \det(A).$$

P4. $\det(A^T) = \det(A)$.

P5. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ denote the row vectors of A. If the ith row vector of A is the sum of two row vectors, say $\mathbf{a}_i = \mathbf{b}_i + \mathbf{c}_i$, then

$$\det(A) = \det(B) + \det(C),$$

where

$$B = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{i-1}, \mathbf{b}_i, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n]^T$$

and

$$C = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{i-1}, \mathbf{c}_i, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n]^T.$$

The corresponding property for columns is also true.

P6. If A has a row (or column) of zeros, then $\det(A) = 0$.

P7. If two rows (or columns) of A are the same, then $\det(A) = 0$.

P8. $\det(AB) = \det(A)\det(B)$.

The first three properties tell us how elementary row operations and elementary column operations performed on a matrix A alter the value of $\det(A)$. They can be very helpful in reducing the amount of work required to evaluate a determinant, since we can use elementary row operations to put several zeros in a row or column of A and then use cofactor expansion along that row or column. We illustrate with an example.

Example 3.4.2 Evaluate

$$\begin{vmatrix} 2 & 1 & 3 & 2 \\ -1 & 1 & -2 & 2 \\ 5 & 1 & -2 & 1 \\ -2 & 3 & 1 & 1 \end{vmatrix}.$$

Solution: Before performing a cofactor expansion, we first use elementary row operations to simplify the determinant:

$$\begin{bmatrix} 2 & 1 & 3 & 2 \\ -1 & 1 & -2 & 2 \\ 5 & 1 & -2 & 1 \\ -2 & 3 & 1 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 0 & 3 & -1 & 6 \\ -1 & 1 & -2 & 2 \\ 0 & 6 & -12 & 11 \\ 0 & 1 & 5 & -3 \end{bmatrix}$$

According to P3, the determinants of the two matrices above are the same. To evaluate the determinant of the matrix on the right, we use cofactor expansion along the first column.

$$\begin{vmatrix} 0 & 3 & -1 & 6 \\ -1 & 1 & -2 & 2 \\ 0 & 6 & -12 & 11 \\ 0 & 1 & 5 & -3 \end{vmatrix} = -(-1)\begin{vmatrix} 3 & -1 & 6 \\ 6 & -12 & 11 \\ 1 & 5 & -3 \end{vmatrix}$$

To evaluate the determinant of the 3×3 matrix on the right, we can use the schematic given in Figure 3.4.1, or, we can continue to use elementary row operations to introduce zeros into the matrix:

$$\begin{vmatrix} 3 & -1 & 6 \\ 6 & -12 & 11 \\ 1 & 5 & -3 \end{vmatrix} \overset{2}{=} \begin{vmatrix} 0 & -16 & 15 \\ 0 & -42 & 29 \\ 1 & 5 & -3 \end{vmatrix} = \begin{vmatrix} -16 & 15 \\ -42 & 29 \end{vmatrix} = 166.$$

Here, we have reduced the 3×3 determinant to a 2×2 determinant by using cofactor expansion along the first column of the 3×3 matrix.

1. $A_{21}(2)$, $A_{23}(5)$, $A_{24}(-2)$ **2.** $A_{31}(-3)$, $A_{32}(-6)$

□

Basic Theoretical Results

The determinant is a useful theoretical tool in linear algebra. We list next the major results that will be needed in the remainder of the text.

1. The volume of the parallelepiped determined by the vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \quad \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

is

$$\boxed{\text{Volume} = |\det(A)|,}$$

where $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$.

2. An $n \times n$ matrix is invertible if and only if $\det(A) \neq 0$.

3. An $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det(A) \neq 0$.

4. An $n \times n$ *homogeneous* linear system $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions if and only if $\det(A) = 0$.

We see, for example, that according to (2), the matrices in Examples 3.4.1 and 3.4.2 are both invertible.

If A is an $n \times n$ matrix with $\det(A) \neq 0$, then the following two methods can be derived for obtaining the inverse of A and for finding the unique solution to the linear system $A\mathbf{x} = \mathbf{b}$, respectively.

1. Adjoint Method for A^{-1}: If A is invertible, then

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A),$$

where adj(A) denotes the transpose of the matrix obtained by replacing each element in A by its cofactor.

2. Cramer's Rule: If $\det(A) \neq 0$, then the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = (x_1, x_2, \dots, x_n)$, where

$$x_k = \frac{\det(B_k)}{\det(A)}, \qquad k = 1, 2, \dots, n,$$

and B_k denotes the matrix obtained when the kth column vector of A is replaced by $\mathbf{b}$.

Example 3.4.3 Use the adjoint method to find A^{-1} if $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 2 \\ 7 & 5 & 8 \end{bmatrix}$.

Solution: We have already shown in Example 3.4.1 that $\det(A) = 41$, so that A is invertible. Replacing each element in A with its cofactor yields the **matrix of cofactors**

$$M_C = \begin{bmatrix} 22 & -10 & -13 \\ 13 & 9 & -17 \\ -6 & -1 & 11 \end{bmatrix},$$

so that

$$\text{adj}(A) = M_C^T = \begin{bmatrix} 22 & 13 & -6 \\ -10 & 9 & -1 \\ -13 & -17 & 11 \end{bmatrix}.$$

Consequently,

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \begin{bmatrix} \frac{22}{41} & \frac{13}{41} & -\frac{6}{41} \\ -\frac{10}{41} & \frac{9}{41} & -\frac{1}{41} \\ -\frac{13}{41} & -\frac{17}{41} & \frac{11}{41} \end{bmatrix}.$$

□

Example 3.4.4 Use Cramer's rule to solve the linear system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 2, \\ 3x_1 + 4x_2 + 2x_3 &= 5, \\ 7x_1 + 5x_2 + 8x_3 &= 3. \end{aligned}$$

Solution: The matrix of coefficients is

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 2 \\ 7 & 5 & 8 \end{bmatrix}.$$

We have already shown in Example 3.4.1 that $\det(A) = 41$. Consequently, Cramer's rule can indeed be applied. In this problem, we have

$$\det(B_1) = \begin{vmatrix} 2 & -1 & 1 \\ 5 & 4 & 2 \\ 3 & 5 & 8 \end{vmatrix} = 91,$$

$$\det(B_2) = \begin{vmatrix} 2 & 2 & 1 \\ 3 & 5 & 2 \\ 7 & 3 & 8 \end{vmatrix} = 22,$$

$$\det(B_3) = \begin{vmatrix} 2 & -1 & 2 \\ 3 & 4 & 5 \\ 7 & 5 & 3 \end{vmatrix} = -78.$$

It therefore follows from Cramer's rule that

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{91}{41}, \qquad x_2 = \frac{\det(B_2)}{\det(A)} = \frac{22}{41}, \qquad x_3 = \frac{\det(B_3)}{\det(A)} = -\frac{78}{41}. \quad \square$$

Exercises for 3.4

Skills

- Be able to compute the determinant of an $n \times n$ matrix.
- Know the effects that elementary row operations and elementary column operations have on the determinant of a matrix.
- Be able to use the determinant to decide if a matrix is invertible.
- Know how the determinant is affected by matrix multiplication and by matrix transpose.
- Be able to compute the adjoint of a matrix and use it to find A^{-1} for an invertible matrix A.

Problems

For Problems 1–7, evaluate the given determinant.

1. $\begin{vmatrix} 5 & -1 \\ 3 & 7 \end{vmatrix}$.

2. $\begin{vmatrix} 3 & 5 & 7 \\ -1 & 2 & 4 \\ 6 & 3 & -2 \end{vmatrix}$.

3. $\begin{vmatrix} 5 & 1 & 4 \\ 6 & 1 & 3 \\ 14 & 2 & 7 \end{vmatrix}$.

4. $\begin{vmatrix} 2.3 & 1.5 & 7.9 \\ 4.2 & 3.3 & 5.1 \\ 6.8 & 3.6 & 5.7 \end{vmatrix}$.

5. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

6. $\begin{vmatrix} 3 & 5 & -1 & 2 \\ 2 & 1 & 5 & 2 \\ 3 & 2 & 5 & 7 \\ 1 & -1 & 2 & 1 \end{vmatrix}$.

7. $\begin{vmatrix} 7 & 1 & 2 & 3 \\ 2 & -2 & 4 & 6 \\ 3 & -1 & 5 & 4 \\ 18 & 9 & 27 & 54 \end{vmatrix}$.

For Problems 8–12, find $\det(A)$. If A is invertible, use the adjoint method to find A^{-1}.

8. $A = \begin{bmatrix} 3 & 5 \\ 2 & 7 \end{bmatrix}$.

9. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.

10. $A = \begin{bmatrix} 3 & 4 & 7 \\ 2 & 6 & 1 \\ 3 & 14 & -1 \end{bmatrix}$.

11. $A = \begin{bmatrix} 2 & 5 & 7 \\ 4 & -3 & 2 \\ 6 & 9 & 11 \end{bmatrix}$.

12. $A = \begin{bmatrix} 5 & -1 & 2 & 1 \\ 3 & -1 & 4 & 5 \\ 1 & -1 & 2 & 1 \\ 5 & 9 & -3 & 2 \end{bmatrix}$.

For Problems 13–17, use Cramer's rule to determine the unique solution to the system $A\mathbf{x} = \mathbf{b}$ for the given matrix and vector.

13. $A = \begin{bmatrix} 3 & 5 \\ 6 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$.

14. $A = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$.

15. $A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix}$.

16. $A = \begin{bmatrix} 5 & 3 & 6 \\ 2 & 4 & -7 \\ 2 & 5 & 9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$.

17. $A = \begin{bmatrix} 3.1 & 3.5 & 7.1 \\ 2.2 & 5.2 & 6.3 \\ 1.4 & 8.1 & 0.9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3.6 \\ 2.5 \\ 9.3 \end{bmatrix}$.

18. If A is an invertible $n \times n$ matrix, prove that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

19. Let A and B be 3×3 matrices with $\det(A) = 3$ and $\det(B) = -4$. Determine

$$\det(2A), \quad \det(A^{-1}), \quad \det(A^T B),$$
$$\det(B^5), \quad \det(B^{-1}AB).$$

3.5 Chapter Review

This chapter has laid out a basic introduction to the theory of determinants.

Determinants and Elementary Row Operations

For a square matrix A, one approach for computing the determinant of A, $\det(A)$, is to use elementary row operations to reduce A to row-echelon form. The effects of the various types of elementary row operations on $\det(A)$ are as follows:

- P_{ij}: permuting two rows of A alters the determinant by a factor of -1.
- $M_i(k)$: multiplying the ith row of A by k multiplies the determinant of the matrix by a factor of k.
- $A_{ij}(k)$: adding a multiple of one row of A to another has no effect whatsoever on $\det(A)$.

A crucial fact in this approach is the following:

Theorem 3.5.1 If A is an $n \times n$ upper (or lower) triangular matrix, its determinant is

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Therefore, since the row-echelon form of A is upper triangular, we can compute $\det(A)$ by using Theorem 3.5.1 and by keeping track of the elementary row operations involved in the row-reduction process.

Cofactor Expansion

Another way to compute $\det(A)$ is via the Cofactor Expansion Theorem: For $n \geq 2$, the determinant of A can be computed using either of the following formulas:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}, \tag{3.5.1}$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}, \tag{3.5.2}$$

where $C_{ij} = (-1)^{i+j}M_{ij}$, and M_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The formulas (3.5.1) and (3.5.2) are referred to as cofactor expansion along the ith row and cofactor expansion along the jth column, respectively. The determinants M_{ij} and C_{ij} are called the **minors** and **cofactors** of A, respectively.

Adjoint Method and Cramer's Rule

If A is an $n \times n$ matrix with $\det(A) \neq 0$, then the following two methods can be derived for obtaining the inverse of A and for finding the unique solution to the linear system $A\mathbf{x} = \mathbf{b}$, respectively.

1. **Adjoint Method for A^{-1}:** If A is invertible, then

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A),$$

where $\mathrm{adj}(A)$ denotes the transpose of the matrix obtained by replacing each element in A by its cofactor.

2. **Cramer's Rule:** If $\det(A) \neq 0$, then the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where

$$x_k = \frac{\det(B_k)}{\det(A)}, \qquad k = 1, 2, \ldots, n,$$

and B_k denotes the matrix obtained when the kth column vector of A is replaced by $\mathbf{b}$.

Additional Problems

For Problems 1–6, evaluate the determinant of the given matrix A by using **(a)** the definition, **(b)** elementary row operations to reduce A to an upper triangular matrix, and **(c)** the Cofactor Expansion Theorem.

1. $A = \begin{bmatrix} -7 & -2 \\ 1 & -5 \end{bmatrix}$.

2. $A = \begin{bmatrix} 6 & 6 \\ -2 & 1 \end{bmatrix}$.

3. $A = \begin{bmatrix} -1 & 4 & 1 \\ 0 & 2 & 2 \\ 2 & 2 & -3 \end{bmatrix}$.

4. $A = \begin{bmatrix} 2 & 3 & -5 \\ -4 & 0 & 2 \\ 6 & -3 & 3 \end{bmatrix}$.

5. $A = \begin{bmatrix} 3 & -1 & -2 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$.

6. $A = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & -5 & 1 \\ 0 & 1 & -4 & 1 \\ -3 & -3 & -3 & -3 \end{bmatrix}$.

For Problems 7–10, suppose that

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \text{ and } \det(A) = 4.$$

Compute the determinant of each matrix below.

7. $\begin{bmatrix} g & h & i \\ -4a & -4b & -4c \\ 2d & 2e & 2f \end{bmatrix}$.

8. $\begin{bmatrix} a-5d & b-5e & c-5f \\ 3g & 3h & 3i \\ -d+3g & -e+3h & -f+3i \end{bmatrix}$.

9. $\begin{bmatrix} 3b & 3e & 3h \\ c-2a & f-2d & i-2g \\ -a & -d & -g \end{bmatrix}$.

10. $3\begin{bmatrix} a-d & b-e & c-f \\ 2g & 2h & 2i \\ -d & -e & -f \end{bmatrix}$.

For Problems 11–14, suppose that A and B are 4×4 invertible matrices. If $\det(A) = -2$ and $\det(B) = 3$, compute each determinant below.

11. $\det(AB)$.

12. $\det(B^2 A^{-1})$.

13. $\det(((A^{-1}B)^T)(2B^{-1}))$.

14. $\det((-A)^3(2B^2))$.

15. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 5 & -2 \\ 4 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 5 \\ 3 & -1 & 4 \\ 2 & -2 & 6 \end{bmatrix}.$$

Determine, if possible,

$\det(A)$, $\det(B)$, $\det(C)$,
$\det(C^T)$, $\det(AB)$, $\det(BA)$,
$\det(B^T A^T)$, $\det(BAC)$, $\det(ACB)$.

16. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}.$$

Use the adjoint method to find B^{-1} and then determine $(A^{-1}B^T)^{-1}$.

For Problems 17–21, use the adjoint method to determine A^{-1} for the given matrix A.

17. $A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 5 & -1 \\ 1 & 1 & 3 \end{bmatrix}$.

18. $A = \begin{bmatrix} 0 & -3 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 1 & 0 & 0 & 5 \end{bmatrix}$.

19. $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & 6 \end{bmatrix}$.

20. $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$.

21. $A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$.

22. Add one row to the matrix

$$A = \begin{bmatrix} 4 & -1 & 0 \\ 5 & 1 & 4 \end{bmatrix}$$

so as to create a 3×3 matrix B with $\det(B) = 10$.

23. True or False: Given any real number r and any 3×3 matrix A whose entries are all nonzero, it is always possible to change at most one entry of A to get a matrix B with $\det(B) = r$.

24. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}$.

(a) Find all value(s) of k for which the matrix A fails to be invertible.

(b) In terms of k, determine the volume of the parallelepiped determined by the row vectors of the matrix A. Is that the same as the volume of the parallelepiped determined by the column vectors of the matrix A? Explain how you know this without any calculation.

25. Repeat the preceding problem for the matrix

$$A = \begin{bmatrix} k+1 & 2 & 1 \\ 0 & 3 & k \\ 1 & 1 & 1 \end{bmatrix}.$$

26. Repeat the preceding problem for the matrix

$$A = \begin{bmatrix} 2 & k-3 & k^2 \\ 2 & 1 & 4 \\ 1 & k & 0 \end{bmatrix}.$$

27. Let A and B be $n \times n$ matrices such that $AB = -BA$. Use determinants to prove that if n is odd, then A and B cannot both be invertible.

28. A real $n \times n$ matrix A is called *orthogonal* if $AA^T = A^T A = I_n$. If A is an orthogonal matrix, prove that $\det(A) = \pm 1$.

For Problems 29– 31, use Cramer's rule to solve the given linear system.

29.
$$\begin{aligned} -3x_1 + x_2 &= 3, \\ x_1 + 2x_2 &= 1. \end{aligned}$$

30.
$$\begin{aligned} 2x_1 - x_2 + x_3 &= 2, \\ 4x_1 + 5x_2 + 3x_3 &= 0, \\ 4x_1 - 3x_2 + 3x_3 &= 2. \end{aligned}$$

31.
$$\begin{aligned} 3x_1 + x_2 + 2x_3 &= -1, \\ 2x_1 - x_2 + x_3 &= -1, \\ 5x_2 + 5x_3 &= -5. \end{aligned}$$

Project: Volume of a Tetrahedron

In this project, we use determinants and vectors to derive the formula for the volume of a tetrahedron with vertices $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$, and $D = (x_4, y_4, z_4)$.

Let h denote the distance from A to the plane determined by B, C, and D. From geometry, the volume of the tetrahedron is given by

$$\text{Volume} = \tfrac{1}{3}h(\text{area of triangle } BCD). \tag{3.5.3}$$

(a) Express the area of triangle BCD in terms of a cross product of vectors.

(b) Use trigonometry to express h in terms of the distance from A to B and the angle between the vector $\overrightarrow{AB}$ and the segment connecting A to the base BCD at a right angle.

(c) Combining (a) and (b) with the volume of the tetrahedron given above, express the volume of the tetrahedron in terms of dot products and cross products of vectors.

(d) Following the proof of part 2 of Theorem 3.1.11, express the volume of the tetrahedron in terms of a determinant with entries in terms of the x_i, y_i, and z_i for $1 \le i \le 4$.

(e) Show that the expression in part (d) is the same as

$$\text{Volume} = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}. \tag{3.5.4}$$

(f) For each set of four points below, determine the volume of the tetrahedron with those points as vertices by using (3.5.3) and by using (3.5.4). Both formulas should yield the same answer.

(i) $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

(ii) $(-1, 1, 2)$, $(0, 3, 3)$, $(1, -1, 2)$, $(0, 0, 1)$.

CHAPTER 4

Vector Spaces

To criticize mathematics for its abstraction is to miss the point entirely. Abstraction is what makes mathematics work. — Ian Stewart

The main aim of this text is to study linear mathematics. In Chapter 2 we studied systems of linear equations, and the theory underlying the solution of a system of linear equations can be considered as a special case of a general mathematical framework for linear problems. To illustrate this framework, we discuss an example.

Consider the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}.$$

It is straightforward to show that this system has solution set

$$S = \{(r - 2s, r, s) : r, s \in \mathbb{R}\}.$$

Geometrically we can interpret each solution as defining the coordinates of a point in space or, equivalently, as the geometric vector with components

$$\mathbf{v} = (r - 2s, r, s).$$

Using the standard operations of vector addition and multiplication of a vector by a real number, it follows that $\mathbf{v}$ can be written in the form

$$\mathbf{v} = r(1, 1, 0) + s(-2, 0, 1).$$

We see that every solution to the given linear problem can be expressed as a linear combination of the two basic solutions (see Figure 4.0.1):

$$\mathbf{v}_1 = (1, 1, 0) \qquad \text{and} \qquad \mathbf{v}_2 = (-2, 0, 1).$$