5. Verify the commutative law of addition for vectors in $\mathbb{R}^4$.

6. Verify the associative law of addition for vectors in $\mathbb{R}^4$.

7. Verify properties (4.1.5)–(4.1.8) for vectors in $\mathbb{R}^3$.

8. Show with examples that if $\mathbf{x}$ is a vector in the first quadrant of $\mathbb{R}^2$ (i.e., both coordinates of $\mathbf{x}$ are positive) and $\mathbf{y}$ is a vector in the third quadrant of $\mathbb{R}^2$ (i.e., both coordinates of $\mathbf{y}$ are negative), then the sum $\mathbf{x}+\mathbf{y}$ could occur in any of the four quadrants.

4.2 Definition of a Vector Space

In the previous section, we showed how the set $\mathbb{R}^n$ of all ordered n-tuples of real numbers, together with the addition and scalar multiplication operations defined on it, has the same algebraic properties as the familiar algebra of geometric vectors. We now push this abstraction one step further and introduce the idea of a vector space. Such an abstraction will enable us to develop a mathematical framework for studying a broad class of linear problems, such as systems of linear equations, linear differential equations, and systems of linear differential equations, which have far-reaching applications in all areas of applied mathematics, science, and engineering.

Let V be a nonempty set. For our purposes, it is useful to call the elements of V vectors and use the usual vector notation $\mathbf{u}, \mathbf{v}, \ldots,$ to denote these elements. For example, if V is the set of all 2×2 matrices, then the vectors in V are 2×2 matrices, whereas if V is the set of all positive integers, then the vectors in V are positive integers. We will be interested only in the case when the set V has an addition operation and a scalar multiplication operation defined on its elements in the following senses:

Vector Addition: A rule for combining any two vectors in V. We will use the usual $+$ sign to denote an addition operation, and the result of adding the vectors $\mathbf{u}$ and $\mathbf{v}$ will be denoted $\mathbf{u}+\mathbf{v}$.

Real (or Complex) Scalar Multiplication: A rule for combining each vector in V with any real (or complex) number. We will use the usual notation $k\mathbf{v}$ to denote the result of scalar multiplying the vector $\mathbf{v}$ by the real (or complex) number k.

To combine the two types of scalar multiplication, we let F denote the set of scalars for which the operation is defined. Thus, for us, F is either the set of all real numbers or the set of all complex numbers. For example, if V is the set of all 2×2 matrices with complex elements and F denotes the set of all complex numbers, then the usual operation of matrix addition is an addition operation on V, and the usual method of multiplying a matrix by a scalar is a scalar multiplication operation on V. Notice that the result of applying either of these operations is always another vector (2×2 matrix) in V.

As a further example, let V be the set of positive integers, and let F be the set of all real numbers. Then the usual operations of addition and multiplication within the real numbers define addition and scalar multiplication operations on V. Note in this case, however, that the scalar multiplication operation, in general, will not yield another vector in V, since when we multiply a positive integer by a real number, the result is not, in general, a positive integer.

We are now in a position to give a precise definition of a vector space.

DEFINITION 4.2.1

Let V be a nonempty set (whose elements are called vectors) on which are defined an addition operation and a scalar multiplication operation with scalars in F. We call V a **vector space over** F, provided the following ten conditions are satisfied:

A1. *Closure under addition:* For each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is also in V. We say that V is **closed under addition**.

A2. *Closure under scalar multiplication:* For each vector $\mathbf{v}$ in V and each scalar k in F, the scalar multiple $k\mathbf{v}$ is also in V. We say that V is **closed under scalar multiplication**.

A3. *Commutativity of addition:* For all $\mathbf{u}, \mathbf{v} \in V$, we have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

A4. *Associativity of addition:* For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, we have

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

A5. *Existence of a zero vector in V:* In V there is a vector, denoted $\mathbf{0}$, satisfying

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \qquad \text{for all } \mathbf{v} \in V.$$

A6. *Existence of additive inverses in V:* For each vector $\mathbf{v}$ In V, there is a vector, denoted $-\mathbf{v}$, in V such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

A7. *Unit property:* For all $\mathbf{v} \in V$,

$$1\mathbf{v} = \mathbf{v}.$$

A8. *Associativity of scalar multiplication:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(rs)\mathbf{v} = r(s\mathbf{v}).$$

A9. *Distributive property of scalar multiplication over vector addition:* For all $\mathbf{u}$, $\mathbf{v} \in V$ and all scalars $r \in F$,

$$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}.$$

A10. *Distributive property of scalar multiplication over scalar addition:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}.$$

Remarks

1. A key point to note is that in order to define a vector space, we must start with all of the following:

   (a) A nonempty set of vectors V.

   (b) A set of scalars F (either $\mathbb{R}$ or $\mathbb{C}$).

(c) An addition operation defined on V.

(d) A scalar multiplication operation defined on V.

Then we must check that the axioms A1–A10 are satisfied.

2. Terminology: A vector space over the real numbers will be referred to as a **real vector space**, whereas a vector space over the complex numbers will be called a **complex vector space**.
3. As indicated in Definition 4.2.1, we will use boldface to denote vectors in a general vector space. In handwriting, it is strongly advised that vectors be denoted either as $\vec{v}$ or as $\underset{\sim}{v}$. This will avoid any confusion between vectors in V and scalars in F.
4. When we deal with a familiar vector space, we will use the usual notation for vectors in the space. For example, as seen below, the set $\mathbb{R}^n$ of ordered n-tuples is a vector space, and we will denote vectors here in the form $(x_1, x_2, \ldots, x_n)$, as in the previous section. As another illustration, it is shown below that the set of all real-valued functions defined on an interval is a vector space, and we will denote the vectors in this vector space by $f, g, \ldots$.

Examples of Vector Spaces

1. The set of all real numbers, together with the usual operations of addition and multiplication, is a real vector space.
2. The set of all complex numbers is a complex vector space when we use the usual operations of addition and multiplication by a complex number. It is also possible to restrict the set of scalars to $\mathbb{R}$, in which case the set of complex numbers becomes a real vector space.
3. The set $\mathbb{R}^n$, together with the operations of addition and scalar multiplication defined in (4.1.13) and (4.1.14), is a real vector space. As we saw in the previous section, the zero vector in $\mathbb{R}^n$ is the n-tuple of zeros $(0, 0, \ldots, 0)$, and the additive inverse of the vector $\mathbf{v} = (x_1, x_2, \ldots, x_n)$ is $-\mathbf{v} = (-x_1, -x_2, \ldots, -x_n)$.

Strictly speaking, for each of the examples above it is necessary to verify all of the axioms A1–A10 of a vector space. However, in these examples, the axioms hold immediately as well-known properties of real and complex numbers and n-tuples.

Example 4.2.2 Let V be the set of all 2×2 matrices with real elements. Show that V, together with the usual operations of matrix addition and multiplication of a matrix by a real number, is a real vector space.

Solution: We must verify the axioms A1–A10. If A and B are in V (that is, A and B are 2×2 matrices with real entries), then $A + B$ and kA are in V for all real numbers k. Consequently, V is closed under addition and scalar multiplication, and therefore Axioms A1 and A2 of the vector space definition hold.

A3. Given two 2×2 matrices

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},$$

we have

$$A + B = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$$
$$= \begin{bmatrix} b_1 + a_1 & b_2 + a_2 \\ b_3 + a_3 & b_4 + a_4 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = B + A.$$

A4. Given three 2×2 matrices

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix},$$

we have

$$\begin{aligned}
(A+B)+C &= \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\right) + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \\
&= \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \\
&= \begin{bmatrix} (a_1+b_1)+c_1 & (a_2+b_2)+c_2 \\ (a_3+b_3)+c_3 & (a_4+b_4)+c_4 \end{bmatrix} \\
&= \begin{bmatrix} a_1+(b_1+c_1) & a_2+(b_2+c_2) \\ a_3+(b_3+c_3) & a_4+(b_4+c_4) \end{bmatrix} \\
&= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1+c_1 & b_2+c_2 \\ b_3+c_3 & b_4+c_4 \end{bmatrix} \\
&= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\right) = A + (B + C).
\end{aligned}$$

A5. If A is any matrix in V, then

$$A + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A.$$

Thus, 0_2 is the zero vector in V.

A6. The additive inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$, since

$$A + (-A) = \begin{bmatrix} a+(-a) & b+(-b) \\ c+(-c) & d+(-d) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2.$$

A7. If A is any matrix in V, then

$$1A = A,$$

thus verifying the unit property.

A8. Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and scalars r and s, we have

$$(rs)A = \begin{bmatrix} (rs)a & (rs)b \\ (rs)c & (rs)d \end{bmatrix} = \begin{bmatrix} r(sa) & r(sb) \\ r(sc) & r(sd) \end{bmatrix} = r\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix} = r(sA),$$

as required.

A9. Given matrices $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ and a scalar r, we have

$$\begin{aligned}
r(A+B) &= r\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\right) \\
&= r\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix} = \begin{bmatrix} r(a_1+b_1) & r(a_2+b_2) \\ r(a_3+b_3) & r(a_4+b_4) \end{bmatrix} \\
&= \begin{bmatrix} ra_1+rb_1 & ra_2+rb_2 \\ ra_3+rb_3 & ra_4+rb_4 \end{bmatrix} = \begin{bmatrix} ra_1 & ra_2 \\ ra_3 & ra_4 \end{bmatrix} + \begin{bmatrix} rb_1 & rb_2 \\ rb_3 & rb_4 \end{bmatrix} = rA + rB.
\end{aligned}$$

A10. Given A, r, and s as in A8 above, we have

$$(r+s)A = \begin{bmatrix} (r+s)a & (r+s)b \\ (r+s)c & (r+s)d \end{bmatrix} = \begin{bmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{bmatrix}$$
$$= \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} + \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix} = rA + sA,$$

as required.

Thus V, together with the given operations, is a real vector space. □

Remark In a manner similar to the previous example, it is easily established that the set of all $m \times n$ matrices with real entries is a real vector space when we use the usual operations of addition of matrices and multiplication of matrices by a real number. We will denote the vector space of all $m \times n$ matrices with real elements by $M_{m\times n}(\mathbb{R})$, and we denote the vector space of all $n \times n$ matrices with real elements by $M_n(\mathbb{R})$.

Example 4.2.3 Let V be the set of all real-valued functions defined on an interval I. Define addition and scalar multiplication in V as follows. If f and g are in V and k is any real number, then $f+g$ and kf are defined by

$$(f+g)(x) = f(x) + g(x) \qquad \text{for all } x \in I,$$
$$(kf)(x) = kf(x) \qquad \text{for all } x \in I.$$

Show that V, together with the given operations of addition and scalar multiplication, is a real vector space.

Solution: It follows from the given definitions of addition and scalar multiplication that if f and g are in V, and k is any real number, then $f+g$ and kf are both real-valued functions on I and are therefore in V. Consequently, the closure axioms A1 and A2 hold. We now check the remaining axioms.

A3. Let f and g be arbitrary functions in V. From the definition of function addition, we have

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x),$$

for all $x \in I$. (The middle step here follows from the fact that $f(x)$ and $g(x)$ are real numbers associated with evaluating f and g at the input x, and real number addition commutes.) Consequently, $f + g = g + f$ (since the values of $f+g$ and $g+f$ agree for every $x \in I$), and so addition in V is commutative.

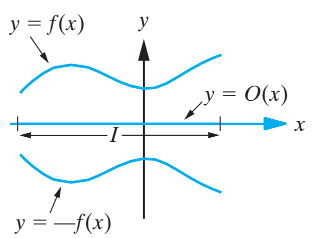

Figure 4.2.1: In the vector space of all functions defined on an interval I, the additive inverse of a function f is obtained by reflecting the graph of f about the x-axis. The zero vector is the zero function $O(x)$.

A4. Let $f, g, h \in V$. Then for all $x \in I$, we have

$$\begin{aligned}[(f+g)+h](x) &= (f+g)(x) + h(x) = [f(x)+g(x)] + h(x) \\ &= f(x) + [g(x)+h(x)] = f(x) + (g+h)(x) \\ &= [f+(g+h)](x).\end{aligned}$$

Consequently, $(f+g)+h = f+(g+h)$, so that addition in V is indeed associative.

A5. If we define the zero function, O, by $O(x) = 0$, for all $x \in I$, then

$$(f+O)(x) = f(x) + O(x) = f(x) + 0 = f(x),$$

for all $f \in V$ and all $x \in I$, which implies that $f + O = f$. Hence, O is the zero vector in V. (See Figure 4.2.1.)

A6. If $f \in V$, then $-f$ is defined by $(-f)(x) = -f(x)$ for all $x \in I$, since

$$[f + (-f)](x) = f(x) + (-f)(x) = f(x) - f(x) = 0$$

for all $x \in I$. This implies that $f + (-f) = O$.

A7. Let $f \in V$. Then, by definition of the scalar multiplication operation, for all $x \in I$, we have

$$(1f)(x) = 1f(x) = f(x).$$

Consequently, $1f = f$.

A8. Let $f \in V$, and let $r, s \in \mathbb{R}$. Then, for all $x \in I$,

$$[(rs)f](x) = (rs)f(x) = r[sf(x)] = r[(sf)(x)].$$

Hence, the functions $(rs)f$ and $r(sf)$ agree on every $x \in I$, and hence $(rs)f = r(sf)$, as required.

A9. Let $f, g \in V$ and let $r \in \mathbb{R}$. Then, for all $x \in I$,

$$\begin{aligned}[r(f+g)](x) &= r[(f+g)(x)] = r[f(x) + g(x)] = rf(x) + rg(x) \\ &= (rf)(x) + (rg)(x) = (rf + rg)(x).\end{aligned}$$

Hence, $r(f + g) = rf + rg$.

A10. Let $f \in V$, and let $r, s \in \mathbb{R}$. Then for all $x \in I$,

$$[(r+s)f](x) = (r+s)f(x) = rf(x)+sf(x) = (rf)(x)+(sf)(x) = (rf+sf)(x),$$

which proves that $(r + s)f = rf + sf$.

Since all parts of Definition 4.2.1 are satisfied, it follows that V, together with the given operations of addition and scalar multiplication, is a real vector space. □

Remark As the previous two examples indicate, a full verification of the vector space definition can be somewhat tedious and lengthy, although it is usually straightforward. Be careful to not leave out any important steps in such a verification.

The Vector Space $\mathbb{C}^n$

We now introduce the most important complex vector space. Let $\mathbb{C}^n$ denote the set of all ordered n-tuples of *complex numbers*. Thus,

$$\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) : z_1, z_2, \ldots, z_n \in \mathbb{C}\}.$$

We refer to the elements of $\mathbb{C}^n$ as **vectors in** $\mathbb{C}^n$. A typical vector in $\mathbb{C}^n$ is $(z_1, z_2, \ldots, z_n)$, where each z_k is a complex number.

Example 4.2.4 The following are examples of vectors in $\mathbb{C}^2$ and $\mathbb{C}^4$, respectively:

$$\mathbf{u} = (2.1 - 3i, -1.5 + 3.9i), \qquad \mathbf{v} = (5 + 7i, 2 - i, 3 + 4i, -9 - 17i).$$ □

In order to obtain a vector space, we must define appropriate operations of "vector addition" and "multiplication by a scalar" on the set of vectors in question. In the case of $\mathbb{C}^n$, we are motivated by the corresponding operations in $\mathbb{R}^n$ and thus define the addition and scalar multiplication operations componentwise. Thus, if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{C}^n$ and k is an arbitrary *complex number*, then

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n),\\ k\mathbf{u} &= (ku_1, ku_2, \ldots, ku_n).\end{aligned}$$

Example 4.2.5 If $\mathbf{u} = (1 - 3i, 2 + 4i)$, $\mathbf{v} = (-2 + 4i, 5 - 6i)$, and $k = 2 + i$, find $\mathbf{u} + k\mathbf{v}$.

Solution: We have

$$\begin{aligned}\mathbf{u} + k\mathbf{v} &= (1 - 3i, 2 + 4i) + (2 + i)(-2 + 4i, 5 - 6i)\\ &= (1 - 3i, 2 + 4i) + (-8 + 6i, 16 - 7i) = (-7 + 3i, 18 - 3i).\end{aligned} \qquad \square$$

It is straightforward to show that $\mathbb{C}^n$, together with the given operations of addition and scalar multiplication, is a *complex* vector space.

Further Properties of Vector Spaces

The main reason for formalizing the definition of an abstract vector space is that any results that we can prove based solely on the definition will then apply to *all* vector spaces we care to examine; that is, we do not have to prove separate results for geometric vectors, $m \times n$ matrices, vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, or real-valued functions, and so on. The next theorem lists some results that can be proved using the vector space axioms.

Theorem 4.2.6 Let V be a vector space over F.

1. The zero vector is unique.
2. $0\mathbf{u} = \mathbf{0}$ for all $\mathbf{u} \in V$.
3. $k\mathbf{0} = \mathbf{0}$ for all scalars $k \in F$.
4. The additive inverse of each element in V is unique.
5. For all $\mathbf{u} \in V$, $-\mathbf{u} = (-1)\mathbf{u}$.
6. If k is a scalar and $\mathbf{u} \in V$ such that $k\mathbf{u} = \mathbf{0}$, then either $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof **1.** Suppose there were two zero vectors in V, denoted $\mathbf{0}_1$ and $\mathbf{0}_2$. Then, for any $\mathbf{v} \in V$, we would have

$$\mathbf{v} + \mathbf{0}_1 = \mathbf{v} \tag{4.2.1}$$

and

$$\mathbf{v} + \mathbf{0}_2 = \mathbf{v}. \tag{4.2.2}$$

We must prove that $\mathbf{0}_1 = \mathbf{0}_2$. But, applying (4.2.1) with $\mathbf{v} = \mathbf{0}_2$, we have

$$\begin{aligned}\mathbf{0}_2 &= \mathbf{0}_2 + \mathbf{0}_1\\ &= \mathbf{0}_1 + \mathbf{0}_2 && \text{(Axiom A3)}\\ &= \mathbf{0}_1 && \text{(from (4.2.2) with } \mathbf{v} = \mathbf{0}_1).\end{aligned}$$

Consequently, $\mathbf{0}_1 = \mathbf{0}_2$, so the zero vector is unique in a vector space.

2. Let $\mathbf{u}$ be an arbitrary element in a vector space V. Since $0 = 0 + 0$, we have

$$0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u},$$

by Axiom A10. Now Axiom A6 implies that the vector $-(0\mathbf{u})$ exists, and adding it to both sides of the previous equation yields

$$0\mathbf{u} + [-(0\mathbf{u})] = (0\mathbf{u} + 0\mathbf{u}) + [-(0\mathbf{u})].$$

Thus, since addition in a vector space is associative (Axiom A4),

$$0\mathbf{u} + [-(0\mathbf{u})] = 0\mathbf{u} + (0\mathbf{u} + [-(0\mathbf{u})]).$$

Applying Axiom A6 on both sides and then using Axiom A5, this becomes

$$\mathbf{0} = 0\mathbf{u} + \mathbf{0} = 0\mathbf{u},$$

and this completes the verification of (2).

3. Using the fact that $\mathbf{0} = \mathbf{0} + \mathbf{0}$ (by Axiom A5), the proof here proceeds along the same lines as the proof of result 2. We leave the verification to the reader as an exercise (Problem 21).

4. Let $\mathbf{u} \in V$ be an arbitrary vector, and suppose that there were two additive inverses, say $\mathbf{v}$ and $\mathbf{w}$, for $\mathbf{u}$. According to Axiom A6, this implies that

$$\mathbf{u} + \mathbf{v} = \mathbf{0} \tag{4.2.3}$$

and

$$\mathbf{u} + \mathbf{w} = \mathbf{0}. \tag{4.2.4}$$

We wish to show that $\mathbf{v} = \mathbf{w}$. Now, Axiom A6 implies that a vector $-\mathbf{v}$ exists, so adding it on the right to both sides of (4.2.3) yields

$$(\mathbf{u} + \mathbf{v}) + (-\mathbf{v}) = \mathbf{0} + (-\mathbf{v}) = -\mathbf{v}.$$

Applying Axioms A4 and A6 on the left side, we simplify this to

$$\mathbf{u} = -\mathbf{v}.$$

Substituting this into (4.2.4) yields

$$-\mathbf{v} + \mathbf{w} = \mathbf{0}.$$

Adding $\mathbf{v}$ to the left of both sides and applying Axioms A4 and A6 once more yields $\mathbf{v} = \mathbf{w}$, as desired.

5. To verify that $-\mathbf{u} = (-1)\mathbf{u}$ for all $\mathbf{u} \in V$, we note that

$$\mathbf{0} = 0\mathbf{u} = (1 + (-1))\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u} = \mathbf{u} + (-1)\mathbf{u},$$

where we have used property 2 and Axioms A10 and A7. The equation above proves that $(-1)\mathbf{u}$ is an additive inverse of $\mathbf{u}$, and by the uniqueness of additive inverses that we just proved, we conclude that $(-1)\mathbf{u} = -\mathbf{u}$, as desired.

Finally, we leave the proof of result 6 in Theorem 4.2.6 as an exercise (Problem 22). ∎

Remark The proof of Theorem 4.2.6 involved a number of tedious and seemingly obvious steps. It is important to remember, however, that in an abstract vector space we are not allowed to rely on past experience in deriving results for the first time. For instance, the statement "$\mathbf{0} + \mathbf{0} = \mathbf{0}$" may seem intuitively clear, but in our newly developed mathematical structure, we must appeal specifically to the rules A1–A10 given for a vector space. Hence, the statement "$\mathbf{0} + \mathbf{0} = \mathbf{0}$" should be viewed as a consequence of Axiom A5 and nothing else. Once we have proved these basic results, of course, then we are free to use them in *any* vector space context where they are needed. This is the whole advantage to working in the general vector space setting.

We end this section with a list of the most important vector spaces that will be required throughout the remainder of the text. In each case the addition and scalar multiplication operations are the usual ones associated with the set of vectors.

- $\mathbb{R}^n$, the (real) vector space of all ordered n-tuples of real numbers.
- $\mathbb{C}^n$, the (complex) vector space of all ordered n-tuples of complex numbers.
- $M_{m\times n}(\mathbb{R})$, the (real) vector space of all $m \times n$ matrices with real elements.
- $M_n(\mathbb{R})$, the (real) vector space of all $n \times n$ matrices with real elements.
- $C^k(I)$, the vector space of all real-valued functions that are continuous and have (at least) k continuous derivatives on I. We will show that this set of vectors is a (real) vector space in the next section.
- P_n, the (real) vector space of all real-valued polynomials of degree $\leq n$ with real coefficients. That is,

$$P_n = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\}.$$

We leave the verification that P_n is a (real) vector space as an exercise (Problem 23).

Exercises for 4.2

Key Terms

Vector space (real or complex), Closure under addition, Closure under scalar multiplication, Commutativity of addition, Associativity of addition, Existence of zero vector, Existence of additive inverses, Unit property, Associativity of scalar multiplication, Distributive properties, Examples: $\mathbb{R}^n$, $\mathbb{C}^n$, $M_n(\mathbb{R})$, $C^k(I)$, P_n.

Skills

- Be able to define a vector space. Specifically, be able to identify and list the ten axioms A1–A10 governing the vector space operations.
- Know each of the standard examples of vector spaces given at the end of the section, and know how to perform the vector operations in these vector spaces.
- Be able to check whether or not each of the axioms A1–A10 holds for specific examples V. This includes, if possible, closure of V under vector addition and scalar multiplication, as well as identification of the zero vector and the additive inverse of each vector in the set V.
- Be able to prove basic properties that hold generally for vector spaces V (see Theorem 4.2.6).

True-False Review

For Questions 1–8, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. The zero vector in a vector space V is unique.

2. If $\mathbf{v}$ is a vector in a vector space V, and r and s are scalars such that $r\mathbf{v} = s\mathbf{v}$, then $r = s$.

3. The set $\mathbb{Z}$ of integers, together with the usual operations of addition and scalar multiplication, forms a vector space.

4. If $\mathbf{x}$ and $\mathbf{y}$ are vectors in a vector space V, then the additive inverse of $\mathbf{x} + \mathbf{y}$ is $(-\mathbf{x}) + (-\mathbf{y})$.

5. The additive inverse of a vector $\mathbf{v}$ in a vector space V is unique.

6. The set $\{0\}$, with the usual operations of addition and scalar multiplication, forms a vector space.

7. The set $\{0, 1\}$, with the usual operations of addition and scalar multiplication, forms a vector space.

8. The set of positive real numbers, with the usual operations of addition and scalar multiplication, forms a vector space.

Problems

For Problems 1–5, determine whether the given set of vectors is closed under addition and closed under scalar multiplication. In each case, take the set of scalars to be the set of all real numbers.

1. The set of all rational numbers.

2. The set of all upper triangular $n \times n$ matrices with real elements.

3. The set of all solutions to the differential equation $y' + 9y = 4x^2$. (Do not solve the differential equation.)

4. The set of all solutions to the differential equation $y' + 9y = 0$. (Do not solve the differential equation.)

5. The set of all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

6. Let
$$S = \{A \in M_2(\mathbb{R}) : \det(A) = 0\}.$$

   (a) Is the zero vector from $M_2(\mathbb{R})$ in S?

   (b) Give an explicit example illustrating that S is not closed under matrix addition.

   (c) Is S closed under scalar multiplication? Justify your answer.

7. Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of all positive integers. Give three reasons why $\mathbb{N}$, together with the usual operations of addition and scalar multiplication, is not a real vector space.

8. We have defined the set $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$, together with the addition and scalar multiplication operations as follows:
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$
$$k(x_1, y_1) = (kx_1, ky_1).$$
Give a complete verification that each of the vector space axioms is satisfied.

9. Determine the zero vector in the vector space $M_{2\times 3}(\mathbb{R})$, and the additive inverse of a general element. (Note that the vector space axioms A1–A4 and A7–A10 follow directly from matrix algebra.)

10. Generalize the previous exercise to find the zero vector and the additive inverse of a general element of $M_{m\times n}(\mathbb{R})$.

11. Let P denote the set of all polynomials whose degree is exactly 2. Is P a vector space? Justify your answer.

12. On $\mathbb{R}^+$, the set of *positive* real numbers, define the operations of addition and scalar multiplication as follows:
$$x + y = xy,$$
$$c \cdot x = x^c.$$
Note that the multiplication and exponentiation appearing on the right side of these formulas refer to the ordinary operations on real numbers. Determine whether $\mathbb{R}^+$, together with these algebraic operations, is a vector space.

13. On $\mathbb{R}^2$, define the operation of addition and multiplication by a real number as follows:
$$(x_1, y_1) + (x_2, y_2) = (x_1 - x_2, y_1 - y_2),$$
$$k(x_1, y_1) = (-kx_1, -ky_1).$$
Which of the axioms for a vector space are satisfied by $\mathbb{R}^2$ with these algebraic operations?

14. On $\mathbb{R}^2$, define the operation of addition by

$$(x_1, y_1) + (x_2, y_2) = (x_1x_2, y_1y_2).$$

Do axioms A5 and A6 in the definition of a vector space hold? Justify your answer.

15. On $M_2(\mathbb{R})$, define the operation of addition by

$$A + B = AB,$$

and use the usual scalar multiplication operation. Determine which axioms for a vector space are satisfied by $M_2(\mathbb{R})$ with the above operations.

16. On $M_2(\mathbb{R})$, define the operations of addition and multiplication by a real number ($\oplus$ and $\cdot$, respectively) as follows:

$$A \oplus B = -(A + B),$$
$$k \cdot A = -kA,$$

where the operations on the right-hand sides of these equations are the usual ones associated with $M_2(\mathbb{R})$.

Determine which of the axioms for a vector space are satisfied by $M_2(\mathbb{R})$ with the operations $\oplus$ and $\cdot$.

For Problems 17–18, verify that the given set of objects together with the usual operations of addition and scalar multiplication is a *complex* vector space.

17. $\mathbb{C}^2$.

18. $M_2(\mathbb{C})$, the set of all 2×2 matrices with complex entries.

19. Is $\mathbb{C}^3$ a *real* vector space? Explain.

20. Is $\mathbb{R}^3$ a *complex* vector space? Explain.

21. Prove part 3 of Theorem 4.2.6.

22. Prove part 6 of Theorem 4.2.6.

23. Prove that P_n is a vector space.

4.3 Subspaces

Let us try to make contact between the abstract vector space idea and the solution of an applied problem. Vector spaces generally arise as the sets containing the unknowns in a given problem. For example, if we are solving a differential equation, then the basic unknown is a function, and therefore any solution to the differential equation will be an element of the vector space V of all functions defined on an appropriate interval. Consequently, the solution set of a differential equation is a subset of V. Similarly, consider the system of linear equations $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix with real elements. The basic unknown in this system, $\mathbf{x}$, is a column n-vector, or equivalently a vector in $\mathbb{R}^n$. Consequently, the solution set to the system is a subset of the vector space $\mathbb{R}^n$. As these examples illustrate, the solution set of an applied problem is generally a subset of vectors from an appropriate vector space (schematically represented in Figure 4.3.1). The question we will need to answer in the future is whether this subset of vectors is a vector space in its own right. The following definition introduces the terminology we will use:

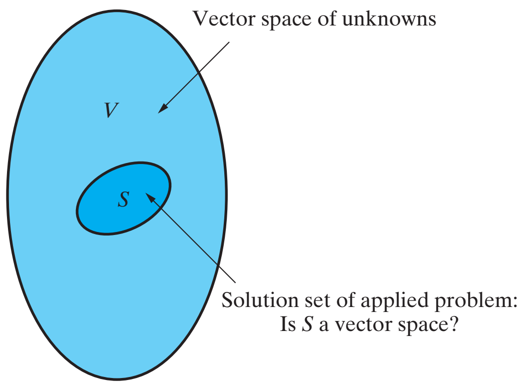

Figure 4.3.1: The solution set S of an applied problem is a subset of the vector space V of unknowns in the problem.