For Problems 7–10, use the ideas in this section to determine a basis for the subspace of $\mathbb{R}^n$ spanned by the given set of vectors.

7. $\{(1, -1, 2), (5, -4, 1), (7, -5, -4)\}$.

8. $\{(1, 3, 3), (1, 5, -1), (2, 7, 4), (1, 4, 1)\}$.

9. $\{(1, 1, -1, 2), (2, 1, 3, -4), (1, 2, -6, 10)\}$.

10. $\{(1, 4, 1, 3), (2, 8, 3, 5), (1, 4, 0, 4), (2, 8, 2, 6)\}$.

11. Let

$$A = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix}.$$

Find a basis for rowspace(A) and colspace(A). Make a sketch to show each subspace in the xy-plane.

12. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 11 & 21 \\ 3 & 7 & 13 \end{bmatrix}$.

(a) Find a basis for rowspace(A) and colspace(A).

(b) Show that rowspace(A) corresponds to the plane with Cartesian equation $2x + y - z = 0$, whereas colspace(A) corresponds to the plane with Cartesian equation $2x - y + z = 0$.

13. Give examples to show how each type of elementary row operation applied to a matrix can change the column space of the matrix.

14. Give an example of a square matrix A whose row space and column space have no nonzero vectors in common.

4.9 The Rank-Nullity Theorem

In Section 4.3, we defined the null space of a real $m \times n$ matrix A to be the set of all real solutions to the associated homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Thus,

$$\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

The dimension of nullspace(A) is referred to as the **nullity** of A and is denoted nullity(A). In order to find nullity(A), we need to determine a basis for nullspace(A). Recall that if rank$(A) = r$, then any row-echelon form of A contains r leading ones, which correspond to the bound variables in the linear system. Thus, there are $n - r$ columns without leading ones, which correspond to free variables in the solution of the system $A\mathbf{x} = \mathbf{0}$. Hence, there are $n - r$ free variables in the solution of the system $A\mathbf{x} = \mathbf{0}$. We might therefore suspect that nullity$(A) = n - r$. Our next theorem, often referred to as the Rank-Nullity Theorem, establishes that this is indeed the case.

Theorem 4.9.1 **(Rank-Nullity Theorem)**

For any $m \times n$ matrix A,

$$\text{rank}(A) + \text{nullity}(A) = n. \tag{4.9.1}$$

Proof If rank$(A) = n$, then by the Invertible Matrix Theorem, the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$. Hence, in this case, nullspace$(A) = \{\mathbf{0}\}$, so nullity$(A) = 0$ and Equation (4.9.1) holds.

Now suppose rank$(A) = r < n$. In this case, there are $n - r > 0$ free variables in the solution to $A\mathbf{x} = \mathbf{0}$. Let $t_1, t_2, \ldots, t_{n-r}$ denote these free variables (chosen as those variables not attached to a leading one in any row-echelon form of A), and let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}$ denote the solutions obtained by sequentially setting each free variable to 1 and the remaining free variables to zero. Note that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ is linearly independent. Moreover, every solution to $A\mathbf{x} = \mathbf{0}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}$:

$$\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_{n-r}\mathbf{x}_{n-r},$$

which shows that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ spans nullspace(A). Thus, $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ is a basis for nullspace(A), and nullity$(A) = n - r$. ■

Example 4.9.2 If

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{bmatrix},$$

find a basis for nullspace(A) and verify Theorem 4.9.1.

Solution: We must find all solutions to $A\mathbf{x} = \mathbf{0}$. Reducing the augmented matrix of this system yields

$$A^{\#} \overset{1}{\sim} \begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -7 & -7 & 0 \\ 0 & -1 & 7 & 7 & 0 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -7 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\boxed{\mathbf{1}.\ \mathrm{A}_{12}(-3),\ \mathrm{A}_{13}(1) \qquad \mathbf{2}.\ \mathrm{A}_{23}(1)}$$

Consequently, there are two free variables, $x_3 = t_1$ and $x_4 = t_2$, so that

$$x_2 = 7t_1 + 7t_2, \qquad x_1 = -9t_1 - 10t_2.$$

Hence,

$$\begin{aligned} \text{nullspace}(A) &= \{(-9t_1 - 10t_2, 7t_1 + 7t_2, t_1, t_2) : t_1, t_2 \in \mathbb{R}\} \\ &= \{t_1(-9, 7, 1, 0) + t_2(-10, 7, 0, 1) : t_1, t_2 \in \mathbb{R}\} \\ &= \text{span}\{(-9, 7, 1, 0), (-10, 7, 0, 1)\}. \end{aligned}$$

Since the two vectors in this spanning set are not proportional, they are linearly independent. Consequently, a basis for nullspace(A) is $\{(-9, 7, 1, 0), (-10, 7, 0, 1)\}$, so that nullity$(A) = 2$. In this problem, A is a 3×4 matrix, and so, in the Rank-Nullity Theorem, $n = 4$. Further, from the foregoing row-echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{0}$, we see that rank$(A) = 2$. Hence,

$$\text{rank}(A) + \text{nullity}(A) = 2 + 2 = 4 = n,$$

and the Rank-Nullity Theorem is verified. □

Systems of Linear Equations

We now examine the linear structure of the solution set to the linear system $A\mathbf{x} = \mathbf{b}$ in terms of the concepts introduced in the last few sections. First we consider the homogeneous case $\mathbf{b} = \mathbf{0}$.

Corollary 4.9.3 Let A be an $m \times n$ matrix, and consider the corresponding homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

1. If rank$(A) = n$, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, so nullspace$(A) = \{\mathbf{0}\}$.
2. If rank$(A) = r < n$, then $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions, all of which can be obtained from

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r}, \tag{4.9.2}$$

where $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ is any linearly independent set of $n - r$ solutions to $A\mathbf{x} = \mathbf{0}$.

Proof Note that part 1 is a restatement of previous results, or can be quickly deduced from the Rank-Nullity Theorem. Now for part 2, assume that $\text{rank}(A) = r < n$. By the Rank-Nullity Theorem, $\text{nullity}(A) = n - r$. Thus, from Theorem 4.6.10, if $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ is any set of $n - r$ linearly independent solutions to $A\mathbf{x} = \mathbf{0}$, it is a basis for $\text{nullspace}(A)$, and so all vectors in $\text{nullspace}(A)$ can be written as

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r},$$

for appropriate values of the constants $c_1, c_2, \ldots, c_{n-r}$. ■

Remark The expression (4.9.2) is referred to as the **general solution** to the system $A\mathbf{x} = \mathbf{0}$.

We now turn our attention to nonhomogeneous linear systems. We begin by formulating Theorem 2.5.9 in terms of $\text{colspace}(A)$.

Theorem 4.9.4 Let A be an $m \times n$ matrix and consider the linear system $A\mathbf{x} = \mathbf{b}$.

1. If $\mathbf{b}$ is not in $\text{colspace}(A)$, then the system is inconsistent.
2. If $\mathbf{b} \in \text{colspace}(A)$, then the system is consistent and has the following:
   - **(a)** a unique solution if and only if $\dim[\text{colspace}(A)] = n$.
   - **(b)** an infinite number of solutions if and only if $\dim[\text{colspace}(A)] < n$.

Proof If we write A in terms of its column vectors as $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$, then the linear system $A\mathbf{x} = \mathbf{b}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}.$$

Consequently, the linear system is consistent if and only if the vector $\mathbf{b}$ is a linear combination of the column vectors of A. Thus, the system is consistent if and only if $\mathbf{b} \in \text{colspace}(A)$. This proves part 1. Parts 2(a) and 2(b) follow directly from Theorem 2.5.9, since $\text{rank}(A) = \dim[\text{colspace}(A)]$. ■

The set of all solutions to a nonhomogeneous linear system is not a vector space, since, for example, it does not contain the zero vector, but the linear structure of $\text{nullspace}(A)$ can be used to determine the general form of the solution of a nonhomogeneous system.

Theorem 4.9.5 Let A be an $m \times n$ matrix. If $\text{rank}(A) = r < n$ and $\mathbf{b} \in \text{colspace}(A)$, then all solutions to $A\mathbf{x} = \mathbf{b}$ are of the form

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r} + \mathbf{x}_p, \tag{4.9.3}$$

where $\mathbf{x}_p$ is any particular solution to $A\mathbf{x} = \mathbf{b}$, and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ is a basis for $\text{nullspace}(A)$.

Proof Since $\mathbf{x}_p$ is a solution to $A\mathbf{x} = \mathbf{b}$, we have

$$A\mathbf{x}_p = \mathbf{b}. \tag{4.9.4}$$

Let $\mathbf{x} = \mathbf{u}$ be an arbitrary solution to $A\mathbf{x} = \mathbf{b}$. Then we also have

$$A\mathbf{u} = \mathbf{b}. \tag{4.9.5}$$

Subtracting (4.9.4) from (4.9.5) yields

$$A\mathbf{u} - A\mathbf{x}_p = \mathbf{0},$$

or equivalently,

$$A(\mathbf{u} - \mathbf{x}_p) = \mathbf{0}.$$

Consequently, the vector $\mathbf{u} - \mathbf{x}_p$ is in nullspace(A), and so there exist scalars $c_1, c_2, \ldots, c_{n-r}$ such that

$$\mathbf{u} - \mathbf{x}_p = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r},$$

since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ is a basis for nullspace(A). Hence,

$$\mathbf{u} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r} + \mathbf{x}_p,$$

as required. ■

Remark The expression given in Equation (4.9.3) is called the **general solution** to $A\mathbf{x} = \mathbf{b}$. It has the structure

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p,$$

where

$$\mathbf{x}_c = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r}$$

is the general solution of the associated homogeneous system and $\mathbf{x}_p$ is one particular solution of the nonhomogeneous system. In later chapters, we will see that this structure is also apparent in the solution of all linear differential equations and in all linear systems of differential equations. It is a result of the linearity inherent in the problem, rather than the specific problem that we are studying. The unifying concept, in addition to the vector space, is the idea of a linear transformation, which we will study in the next chapter.

Example 4.9.6 Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 10 \\ -4 \end{bmatrix}.$$

Verify that $\mathbf{x}_p = (1, 1, -1, 1)$ is a particular solution to $A\mathbf{x} = \mathbf{b}$, and use Theorem 4.9.5 to determine the general solution to the system.

Solution: For the given $\mathbf{x}_p$, we have

$$A\mathbf{x}_p = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ -4 \end{bmatrix} = \mathbf{b}.$$

Consequently, $\mathbf{x}_p = (1, 1, -1, 1)$ is a particular solution to $A\mathbf{x} = \mathbf{b}$. Further, from Example 4.9.2, a basis for nullspace(A) is $\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = (-9, 7, 1, 0)$ and $\mathbf{x}_2 = (-10, 7, 0, 1)$. Thus, the general solution to $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x}_c = c_1\mathbf{x}_1 + c_2\mathbf{x}_2,$$

and therefore, from Theorem 4.9.5, the general solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \mathbf{x}_p = c_1(-9, 7, 1, 0) + c_2(-10, 7, 0, 1) + (1, 1, -1, 1),$$

which can be written as

$$\mathbf{x} = (-9c_1 - 10c_2 + 1, 7c_1 + 7c_2 + 1, c_1 - 1, c_2 + 1). \qquad \square$$

Exercises for 4.9

Skills

- For a given matrix A, be able to determine the rank from the nullity, or the nullity from the rank.
- Know the relationship between the rank of a matrix A and the consistency of a linear system $A\mathbf{x} = \mathbf{b}$.
- Know the relationship between the column space of a matrix A and the consistency of a linear system $A\mathbf{x} = \mathbf{b}$.
- Be able to formulate the solution set to a linear system $A\mathbf{x} = \mathbf{b}$ in terms of the solution set to the corresponding homogeneous linear equation.

True-False Review

For Questions 1–9, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. For an $m \times n$ matrix A, the nullity of A must be at least $|m - n|$.

2. If A is a 7×9 matrix with $\text{nullity}(A) = 2$, then $\text{rowspace}(A) = \mathbb{R}^7$.

3. If A is a 9×7 matrix with $\text{nullity}(A) = 0$, then $\text{rowspace}(A) = \mathbb{R}^7$.

4. The nullity of an $n \times n$ *upper triangular* matrix A is simply the number of zeros appearing on the main diagonal of A.

5. An $n \times n$ matrix A for which $\text{nullspace}(A) = \text{colspace}(A)$ cannot be invertible.

6. For all $m \times n$ matrices A and B, $\text{nullity}(A + B) = \text{nullity}(A) + \text{nullity}(B)$.

7. For all $n \times n$ matrices A and B, $\text{nullity}(AB) = \text{nullity}(A) \cdot \text{nullity}(B)$.

8. For all $n \times n$ matrices A and B, $\text{nullity}(AB) \geq \text{nullity}(B)$.

9. If $\mathbf{x}_p$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{y} + \mathbf{x}_p$ is also a solution for any $\mathbf{y}$ in $\text{nullspace}(A)$.

Problems

For Problems 1–4, determine the null space of A and verify the Rank-Nullity Theorem.

1. $A = \begin{bmatrix} 1 & 0 & -6 & -1 \end{bmatrix}$.

2. $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$.

3. $A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & 4 \\ 1 & 1 & 0 \end{bmatrix}$.

4. $A = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 2 & 9 & -1 & 7 \\ 2 & 8 & -2 & 6 \end{bmatrix}$.

For Problems 5–8, determine the nullity of A "by inspection" by appealing to the Rank-Nullity Theorem. Avoid computations.

5. $A = \begin{bmatrix} 2 & -3 \\ 0 & 0 \\ -4 & 6 \\ 22 & -33 \end{bmatrix}$.

6. $A = \begin{bmatrix} 1 & 3 & -3 & 2 & 5 \\ -4 & -12 & 12 & -8 & -20 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & -3 & 2 & 6 \end{bmatrix}$.

7. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

8. $A = \begin{bmatrix} 0 & 0 & 0 & -2 \end{bmatrix}$.

For Problems 9–12, determine the solution set to $A\mathbf{x} = \mathbf{b}$, and show that all solutions are of the form (4.9.3).

9. $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 9 \\ 1 & 5 & 21 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 11 \\ 10 \end{bmatrix}$.

10. $A = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & 5 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 13 \end{bmatrix}$.

11. $A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & -7 \\ 1 & 1 & 1 \\ 2 & 2 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ 0 \\ -6 \end{bmatrix}$.

12. $A = \begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 2 & -1 & 7 \\ 4 & 2 & -3 & 13 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

13. Show that a 3×7 matrix A with $\text{nullity}(A) = 4$ must have $\text{colspace}(A) = \mathbb{R}^3$. Is $\text{rowspace}(A) = \mathbb{R}^3$?

14. Show that a 6×4 matrix A with $\text{nullity}(A) = 0$ must have $\text{rowspace}(A) = \mathbb{R}^4$. Is $\text{colspace}(A) = \mathbb{R}^4$?

15. Prove that if $\text{rowspace}(A) = \text{nullspace}(A)$, then A contains an even number of columns.

16. Show that a 5×7 matrix A must have $2 \leq \text{nullity}(A) \leq 7$. Give an example of a 5×7 matrix A with $\text{nullity}(A) = 2$ and an example of a 5×7 matrix A with $\text{nullity}(A) = 7$.

17. Show that 3×8 matrix A must have $5 \leq \text{nullity}(A) \leq 8$. Give an example of a 3×8 matrix A with $\text{nullity}(A) = 5$ and an example of a 3×8 matrix A with $\text{nullity}(A) = 8$.

18. Prove that if A and B are $n \times n$ matrices and A is invertible, then

$$\text{nullity}(AB) = \text{nullity}(B).$$

[**Hint:** $B\mathbf{x} = \mathbf{0}$ if and only if $AB\mathbf{x} = \mathbf{0}$.]

4.10 The Invertible Matrix Theorem II

In Section 2.8, we gave a list of characterizations of invertible matrices (Theorem 2.8.1). In view of the concepts introduced in this chapter, we are now in a position to add to the list that was begun there.

Theorem 4.10.1 **(Invertible Matrix Theorem)**

Let A be an $n \times n$ matrix with real elements. The following conditions on A are equivalent:

(a) A is invertible.

(h) $\text{nullity}(A) = 0$.

(i) $\text{nullspace}(A) = \{\mathbf{0}\}$.

(j) The columns of A form a linearly independent set of vectors in $\mathbb{R}^n$.

(k) $\text{colspace}(A) = \mathbb{R}^n$ (that is, the columns of A span $\mathbb{R}^n$).

(l) The columns of A form a basis for $\mathbb{R}^n$.

(m) The rows of A form a linearly independent set of vectors in $\mathbb{R}^n$.

(n) $\text{rowspace}(A) = \mathbb{R}^n$ (that is, the rows of A span $\mathbb{R}^n$).

(o) The rows of A form a basis for $\mathbb{R}^n$.

(p) A^T is invertible.

Proof The equivalence of (a) and (h) follows at once from Theorem 2.8.1(d) and the Rank-Nullity Theorem (Theorem 4.9.1). The equivalence of (h) and (i) is immediately clear. The equivalence of (a) and (j) is immediate from Theorem 2.8.1(c) and Theorem 4.5.14. Since the dimension of $\text{colspace}(A)$ is simply $\text{rank}(A)$, the equivalence of (a) and (k) is immediate from Theorem 2.8.1(d). Next, from the definition of a basis,