Purdue University **January 3, 2024**
MA 544

Final Exam

PUID:______________________

a) Legibly print your PUID above.

b) Do not open this test booklet until you are directed to do so.

c) This test is closed book and closed notes. You may not use a calculator, cell phone, ipad, computer, or any other electronic device during the exam.

d) You will have 120 min. to complete the exam.

e) Do not spend too much time on any one problem. Read them all through first and attack them in the order that allows you to make the most progress. Repeat: Budget your time wisely!

f) Throughout the test, **show your work so that your reasoning is clear**. Otherwise no credit will be given.

g) If you need extra room, use the back of the pages or the blank pages at the end of the exam. Just make sure I can follow your work.

Problem	Points	Grade
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
Total	100	

1 (15 pts)**.** A sequence of measurable functions $\{f_n\}_{n\geq 1}$ on a measure space $(X, \mathcal{F}, \mu)$ is said to to **uniformly integrable** with respect to μ if

$$\lim_{M\to\infty} \left(\sup_{n\geq 1} \int_{\{|f_n|\geq M\}} |f_n| \, d\mu \right) = 0.$$

Prove that if $\{f_n\}_{n\geq 1}$ is uniformly integrable with respect to μ, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mu(A) < \delta$ then $\int_A |f_n| \, d\mu < \varepsilon$ for all $n \geq 1$.

2 (15 pts)**.** Let

$$F(t) = \int_0^1 \sqrt{x}\sin(tx)\,dx, \qquad \forall t \in \mathbb{R}.$$

Prove that $F'(t) = \int_0^1 x^{3/2}\cos(tx)\,dx$ for every $t \geq \mathbb{R}$. (Make sure to justify all of your steps.)

3 (20 pts)**.** For this problem, we are considering functions on $\mathbb{R}^n$ with Lebesgue measure.

a) Let f_n be a sequence of functions such that $\sup_n \|f_n\|_\infty \leq M$ for some $M < \infty$. If f is a function such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$ for some $p > 0$, then prove that $\|f\|_\infty \leq M$ also.

(Continued on the next page...)

b) Give an example of a sequence of functions with $\inf_n \|f_n\|_\infty \geq M$ for some $M > 0$ and a function f such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$ for all $p > 0$, but for which $\|f\|_\infty < M$.

4 (15 pts)**.** Assume that $f, g \in L^1(\mathbb{R})$ are such that the functions $xf(x)$ and $xg(x)$ are also integrable. Prove that the function $x(f * g)(x)$ is also integrable and that

$$\int x(f * g)(x)\, dx = \iint (x + y) f(x) g(y)\, (dx\, dy)$$

5 (15 pts)**.** Suppose that ϕ is convex on (a, b) and continuous on $[a, b]$. Prove that

$$(b-a)\phi\left(\frac{a+b}{2}\right) \leq \int_a^b \phi(x)\,dx \leq \frac{b-a}{2}(\phi(a)+\phi(b)).$$

6 (20 pts)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\mathcal{G}$ be a σ-algebra on another space Y. Assume that $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is measurable (that is $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{G}$), and then define the measure $\nu : \mathcal{G} \to [0, \infty]$ by

$$\nu(A) = \mu(f^{-1}(A)).$$

a) Prove that ν is a measure on $(Y, \mathcal{G})$.

(Continued on the next page...)

b) Prove that if g is a non-negative measureable function on the measure space $(Y, \mathcal{G}, \nu)$, then

$$\int_Y g \, d\nu = \int_X (g \circ f) \, d\mu.$$

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