QUALIFYING EXAM
August 2024
Math 544

Instructions: There are a total of 8 problems. Problems 1 and 2 are worth 10 points each. Problems 3-8 are worth 20 points each. For each problem, use the space provided for its solution including the extra page if needed.

For Legesgue measurable functions f on $\mathbb{R}^n$ we simply write $\int_{\mathbb{R}^n} f(x)dx$ instead of $\int_{\mathbb{R}^n} f dm$. For an interval $[a,b]$ with its Lebesgue measure m we simply write $\int_a^b f(x)dx$ for $\int_{[a,b]} f dm$, and similarly for open or half open intervals.

Write your solutions to each problem in clear, concise and correct English. Solutions must contain full details and should be presented clearly so that the grader can follow your argument.

Problem 1–(10 pts)

(i) **(5 pts)** Define, carefully, what it means for a function $f : [0, 1] \to \mathbb{R}$ to be of bounded variation (BV).

(ii) **(5 pts)** Let f and f_n, $n = 1, 2 \ldots$ be in $BV[0, 1]$ and suppose that $V(f_n - f) \to 0$. Prove that there is a subsequence f_{n_k} such that $f'_{n_k} \to f'$, a.e.

Problem 2–(10 pts) Let $\{f_n\}$ and $\{g_n\}$ be two sequences of real valued measurable functions defined on the measure space $(X, \mathcal{F}, \mu)$ with the property that

$$\sum_{n=1}^{\infty} \mu\{x \in X : f_n(x) \neq g_n(x)\} < \infty.$$

Prove that $\sum_{n=1}^{\infty} (f_n - g_n)$ converges a.e.

Problem 3–(20 pts) Let $f \in L^p(0,\infty)$, $1 < p < \infty$. Prove that there exist a sequence $\{x_n\}$, $x_n \to \infty$, such that $\lim_{n\to\infty} x_n^{1/p}|f(x_n)| = 0$.

Problem 4–(20 pts) Let f be a nonnegative real valued function on the measure space $(X, \mathcal{F}, \mu)$ with the property that $\mu\{x \in X : 0 \leq f(x) < 2\} < \infty$ and such that

$$\int_E f d\mu \leq (\mu(E))^{\frac{1}{q}}, \quad 1 < q < \infty,$$

for all sets $E \in \mathcal{F}$. Let $1 < p < \infty$ be the conjugate exponent of q. That is, $1/p + 1/q = 1$. Prove that $f \in L^r(\mu)$ for any $1 \leq r < p$.

Hint: Start by expressing X as the disjoint union of sets depending on the function f.

(Extra page for work on problem 4 as needed.)

Problem 5–(20 pts) Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of Lebesgue measurable functions on $\mathbb{R}^n$. Assume $f_1 \in L^p$, $0 < p < \infty$, and that

$$m\{x \in \mathbb{R}^n : |f_j(x)| > \lambda\} \leq m\{x \in \mathbb{R}^n : |f_1(x)| > \lambda\},$$

for all j and all $\lambda > 0$. Prove that

$$\sum_{k=3}^{\infty} \frac{1}{k^2 \log(k))} \int_{\mathbb{R}} \left(\max_{1 \leq j \leq k} |f_j| \right)^p dm < \infty.$$

(Extra page for work on problem 5 as needed.)

Problem 6–(20 pts) For any two sets A and B define $A+B=\{x+y : x\in A \text{ and } y\in B\}$ and $-A=\{-x : x\in A\}$. Let $A\subset R$ be Lebesgue measurable with $0<m(A)<\infty$. Prove that $A-A=A+(-A)$ contains and interval of the form $(-a,a)$ for some positive a.

Hint: Define $F(x)=m\left((x+A)\cap A\right)$. Writing it as a convolution, explore continuity ...

(Extra page for work on problem 6 as needed.)

Problem 7–(20 pts) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. The sequence of measurable functions $\{f_n\}$ is said to be uniformly integrable (UI) if

$$\lim_{\lambda \to \infty} \sup_n \int_{\{|f_n| > \lambda\}} |f_n| d\mu = 0.$$

Prove that $\{f_n\}$ is (UI) if and only if the following two conditions hold: (i) $\sup_n \|f_n\|_1 < \infty$ and (ii) given $\varepsilon > 0$ there is a $\delta > 0$ such that $\sup_n \int_E |f_n| d\mu < \varepsilon$, whenever $\mu(E) < \delta$.

(Extra page for work on problem 7 as needed.)

Problem 8–(20–pts) Let $X = (0,2)$ and $Y = (0,1)$ with the Lebesgue measure. Consider the measurable function $F : X \times Y \to \mathbb{R}$ defined by

$$F(x,y) = \frac{xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

(i) Verify that

$$\int_0^2 \left(\int_0^1 F(x,y)dy \right) dx \neq \int_0^1 \left(\int_0^2 F(x,y)dx \right) dy$$

(ii) Explain why Fubini's Theorem does not apply.

(Extra page for work on problem 8 as needed.)