Qualifying Examination

MA 553

January 4, 2024

Time: 2 hours

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2	
3	
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Total	

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(20 pts) 1). Let p, q and r be three distinct prime numbers with $p > qr$. Let n be a positive integer. Show that every group G of order $O(G) = p^n qr$ is solvable. Conclude that every group of order 294 or 1210 is solvable.

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2). Let q be a prime number and let

$$f_q(x) = x^{q-1} + x^{q-2} + \cdots + 1$$

(20 pts) a) Suppose a prime number p divides $f_q(a)$ for some integer a. Prove that either $p = q$ or $p \equiv 1 (\text{mod } q)$.

(10 pts) b) Prove there are infinitely many primes of the form $qb + 1$, b an integer.

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3). (10 pts) a) Prove that $A = \mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

(20 pts) b) Show that

$$A/(3 + 2\sqrt{-2}) \simeq \mathbb{Z}/17\mathbb{Z} := \mathbb{F}_{17}.$$

(15 pts) c) Show that $x^4 + 3$ is irreducible over $\mathbb{F}_{17}$ and conclude that

$$f(x) = x^4 - 170x + 9 + 4\sqrt{-2} \in A[x]$$

is irreducible over $A[x]$.

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4). Let n be an integer and let

$$f(x) = x^3 - (n-3)x^2 - nx - 1.$$

(10 pts) a) Show that $f(x)$ is irreducible over $\mathbb{Q}[x]$.

(15 pts) b) Show that if a is a root of $f(x)$, then $-1/(a+1)$ is also a root of $f(x)$.

(10 pts) c) Let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order 3. Conclude that all the roots of $f(x) = 0$ are real.

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5). Let $f(x) = (x^2 - 5)(x^3 - 2)$. Let K be the splitting field of $f(x)$ over $\mathbb{Q}$.

(10 pts) a) Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

(15 pts) b) Determine all subfields L of K such that $L/\mathbb{Q}$ is of degree 2.

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6).

(10 pts) a) Prove that $x^4 + 1$ is irreducible over $\mathbb{Z}[x]$ using Eisenstein Criterion.

(10 pts) b) Let ζ_8 be a primitive 8th root of 1. Compute $[\mathbb{Q}(\zeta_8) : \mathbb{Q}]$.

(10 pts) c) Show that $\mathbb{Q}(\zeta_8)/\mathbb{Q}$ is Galois and determine the Galois group through the map

$$\pi : \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \longrightarrow \mathbb{Z}/8\mathbb{Z}$$

$$\sigma \longmapsto \pi(\sigma),$$

where $\sigma(\zeta_8) = \zeta_8^{\pi(\sigma)}$.

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7).

(15 pts) a) Using that $\mathbb{Z}[\sqrt{-2}]$ is a unique factorization domain show that

$$f(x) = x^3 - (1 + \sqrt{-2})$$

is irreducible over $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Q}(\sqrt{-2})$.

(10 pts) b) Determine the Galois group of $f(x)$ over $\mathbb{Q}(\sqrt{-2})$.

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