Qualifying Examination
MA 553
August 8, 2024
Time: 2 hours

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2	
3	
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5	
6	
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Total	

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1).

(15 pts) a) Show that a non-abelian group G has an abelian quotient G/Z, Z the center of G, if and only if $[G : G] \subset Z$. Here $[G : G]$ is the commutatant of G (derived group of G). Give an example of such a group.

(5 pts) b) Let G be a non-abelian group with a trivial center. Denote by $\mathrm{Aut}(G)$, the group of automorphisms of G. Show that $\mathrm{Aut}(G)$ is not abelian.

(5 pts) c) Assume $\mathrm{Aut}(G)$ is solvable when G is any group. Show that G is solvable.

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2).

(5 pts) a) Show that $f(x) = x^3 - x - 1$ is irreducible over $\mathbb{Z}$.

(10 pts) b) Show that $x+1$ and x^3-x-1 are relatively prime in $\mathbb{Z}[x]$; i.e., the ideal generated by them is $\mathbb{Z}[x]$.

(15 pts) c) Give a simpler description of the ring

$$\mathbb{Z}[x]/((x+1)(x^3-x-1))$$

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3). Let $n \geq 1$ be a positive integer. Set

$$f_n(x) = (x-1)(x-2)\dots(x-n) - 1.$$

(20 pts) a) Show that $f_n(x)$ is irreducible over $\mathbb{Z}$ for all $n \geq 1$. Is it irreducible over $\mathbb{Q}$? Why?

(10 pts) b) Let p and m be two positive integers. Assume p is a prime. Let α be a root of $f_p(x)$ and β one of $f_m(x)$. Let $K = Q(\alpha, \beta)$. What are possible values of $[K : \mathbb{Q}]$? Give reasons.

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4).

(10 pts) a) Let R be a unique factorization domain and let

$$f(x,y) = x^9 - yx^7 + yx^3 + 7yx + y \in R[x,y]$$

show that $f(x,y)$ is irreducible over $R[x,y]$.

(10 pts) b) Let $K = F(x^9/x^7 - x^3 - 7x - 1)$, where F is a field. Assume F is the field of quotients of $R[y]$. Determine $[F(x) : K]$.

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5).

(10 pts) a) Show that $x^4 + 1$ is irreducible over $\mathbb{Z}[x]$ using Eisenstein Criterion.

(20 pts) b) Show that $x^4 + 1 \in \mathbb{Z}[x]$ is reducible modulo every prime p. (Hint: For odd p, $x^8 - 1$ divides $x^{p^2} - x$.). Thus a polynomial could be irreducible over $\mathbb{Z}$ while it is reducible modulo every prime.

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6).

(15 pts) a) Show that $f(x) = x^3 - 3x - 1$ is irreducible over $\mathbb{Q}$ and determine its Galois group. Does $f(x)$ have any complex roots? Justify your answer.

(Hint: For $f(x) = x^3 + px + q, \quad \Delta = -4p^3 - 27q^2$.)

(15 pts) b) Show that $g(x) = x^3 - 4x + 2$ is irreducible over $\mathbb{Q}$ and its Galois group over $\mathbb{Q}$ is isomorphic to S_3, but all its roots are real (prove this). Thus having only real roots is not enough to determine the Galois group of an irreducible cubic over $\mathbb{Q}$.

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7).

(8 pts) a) Using that $\mathbb{Z}[\sqrt{-1}]$ is a unique factorization domain (UFD) show that

$$x^3 - (1 + \sqrt{-1})$$

is irreducible over $\mathbb{Z}[\sqrt{-1}]$ and $\mathbb{Q}(\sqrt{-1})$.

(7 pts) b) Show that the polynomial $f(x) := x^6 - 2x^3 + 2$ is irreducible over $\mathbb{Q}$ which has $\alpha = \sqrt[3]{1 + \sqrt{-1}}$ and $\beta = \sqrt[3]{1 - \sqrt{-1}}$ among its roots. What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$?

(10 pts) c) Determine the irreducible polynomial for a primitive 12th root of unity (12th cyclotomic polynomial).

(10 pts) d) Let $L = \mathbb{Q}(\alpha, \beta)$. Show that $\sqrt[3]{2} \in L$. Using part c) prove that $\sqrt[3]{2} \notin \mathbb{Q}(\alpha)$. What is $[L : \mathbb{Q}]$?

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