554 QUALIFYING EXAM, JAN 5, 2024

Attempt all questions. Time 2 hrs.

1. (5 + 10 pts)
   (a) Define the rank of a linear mapping between two vector spaces.
   (b) Let E, F, G, H be four finite dimensional vector spaces over a field k and $u : E \longrightarrow F, v : F \longrightarrow G, w : G \longrightarrow H$ be linear mappings. Prove that
   $$\mathrm{rk}(v \circ u) + \mathrm{rk}(w \circ v) \leq \mathrm{rk}(v) + \mathrm{rk}(w \circ v \circ u).$$
2. (5 + 10 pts)
   (a) State (but do not prove) the additive Jordan decomposition theorem for an endomorphism of a finite dimensional complex vector space.
   (b) Let V be a finite dimensional complex vector space and $u, v \in \mathrm{End}(V)$ such that $u \circ v = v \circ u$. Express the Jordan decomposition of $u \circ v$ in terms of the Jordan decompositions of u and v. Justify your answer.
3. (5 + 5 pts) Let V be a finite dimensional complex inner product space and $u \in \mathrm{End}(V)$ and W a subspace of V closed under u.
   (a) Prove that $W^{\perp}$ is closed under the adjoint of u.
   (b) Suppose that u is normal. Prove that $W^{\perp}$ is closed under u.
4. (5 + 10 pts) Let V be a finite dimensional complex inner product space and $u \in \mathrm{End}(V)$.
   (a) Prove that there exist a unique pair of self-adjoint endomorphisms $v, w \in \mathrm{End}(V)$ such that $u = v + \sqrt{-1}w$.
   (b) Prove that a normal endomorphism is self-adjoint if and only if its eigenvalues are all real.
5. (5 + 5 pts) Let X be the complex matrix
   $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix}.$$
   Determine the rational canonical form and the Jordan canonical form of X.
6. (5 + 5 pts)
   (a) When are two $n \times n$ complex matrices similar ?
   (b) Let A be an $n \times n$ complex matrix with characteristic polynomial $(X - 1)^n$. Prove that A is invertible and that A is similar to A^{-1}.
7. (5 + 10 pts) Let V be a finite dimensional complex vector space and $u \in \mathrm{End}(V)$.
   (a) Define the minimal polynomial of u.
   (b) Suppose that $u^k = \mathrm{Id}_V$ for some positive integer k. Prove that u is diagonalizable.
8. (10 pts) Let V be an n-dimensional complex inner product space and let $u, v \in \mathrm{End}(V)$ be two unitary transformations. Prove that $|\det(u + v)| \leq 2^n$.