554 QUALIFYING EXAM, AUG 9, 2024

Attempt all questions. Time 2 hrs.

1. (5 + 5 + 10 pts) Let V be a n-dimensional vector space over a field $\mathbb{F}$.
   (a) Define the rank and the nullity of an endomorphism of V.
   (b) If $\phi \in \mathrm{End}(V)$, prove that $\mathrm{rank}(\phi) + \mathrm{nullity}(\phi) = n$.
   (c) If $\phi, \psi \in \mathrm{End}(V)$, then prove that
   $$\mathrm{rank}(\phi) + \mathrm{rank}(\psi) - n \leq \mathrm{rank}(\phi \circ \psi) \leq \min(\mathrm{rank}(\phi), \mathrm{rank}(\psi)).$$
2. (10 pts) Let V be a finite dimensional complex vector space and $\phi \in \mathrm{End}(V)$. Suppose $\alpha_1, \ldots, \alpha_m$ are eigenvectors belonging to distinct eigenvalues of ϕ. Prove that $\alpha_1, \ldots, \alpha_m$ are linearly independent.
3. (5 + 5 pts)
   (a) When are two $n \times n$ complex matrices similar ?
   (b) Can an $n \times n$ complex matrix A be similar to the matrix $A + \mathrm{Id}_n$?
4. (10 pts) Let $n \geq 2$, and A be an $n \times n$ complex matrix such that A is a Jordan block with 0's on the diagonal. What is the Jordan canonical form of A^2 ?
5. (10 pts) Let V be an n-dimensional complex inner product space. Suppose $A = \{\alpha_1, \ldots, \alpha_n\}$ is a basis for V. Prove that there exists a unique basis $B = \{\beta_1, \ldots, \beta_n\}$ such that $(\alpha_i, \beta_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. (Here δ_{ij} is the Kronecker symbol.)
6. (5+5+10pts) Let V be a finite dimensional complex inner product vector space and $u \in \mathrm{End}(V)$.
   (a) Define the adjoint u^* of u ?
   (b) What does it mean to say that u is normal ?
   (c) Prove that for u to be normal it is necessary and sufficient that every subspace of V that is closed under u, is also closed under u^*.
7. (20 pts) Let V be the real vector space of polynomials of real polynomials in X of degree at most 3 equipped with the inner product given by
   $$(f, g) = \int_0^1 f(X)g(X)dX.$$
   Calculate the distance from the origin to the set of monic polynomials (i.e. polynomials with leading coefficient equal to 1).