Name: ______________________________ **PUID:** ________________________

Your TA's Name: ________________________ **Recitation Time:** ______________

MA 16100

SPRING 2024

EXAM 1 SOLUTIONS

The problems are numbered 1-10.

For problems 1-8 indicate your answer by filling in the appropriate circle next to the answer choice. Partial credit will **not** be awarded for problems 1-8.

This exam is out of 100 points. Problems 1-8 are worth 8 points each and problems 9 and 10 are worth 16 points each. You will receive 4 points for signing the bottom of this page.

Extra scratch paper is not permitted. Write all your work in this exam booklet.

Write your name and PUID on each page. This will help us locate and successfully grade your test if the pages become separated.

You may not leave the room before 20 minutes have passed. If you finish the exam between when 20 and 50 minutes have passed, you may leave the room after turning in the exam booklet. If you finish within the last 10 minutes of the exam, you MUST REMAIN SEATED until your TA comes and collects your exam booklet.

Exam Policies:

1. Students may not open the exam until instructed to do so.
2. Students must obey the orders and requests by all proctors, TAs, and the instructor.
3. No student may leave in the first 20 min or in the last 10 min of the exam.
4. Books, notes, calculators, or any electronic devices are not allowed on the exam, and they should not even be in sight in the exam room. Students may not look at anybody else's test, and may not communicate with anybody else except, if they have a question, with their TA or the instructor.
5. After time is called, the students have to put down all writing instruments and remain in their seats, while the TAs will collect the exams.
6. Any violation of these rules and any act of academic dishonesty may result in severe penalties. Additionally, all violators will be reported to the Office of the Dean of Students.

I have read and understand the exam rules stated above:

Student Signature: ____

Name: ______________________________ **PUID:** ____________________

1. (**8 Points**) If $\log_{27}(x) = 3$ compute $\log_3(x^4)$.

★ **A** 36.
◯ **B** 12.
◯ **C** 6.
◯ **D** 9.
◯ **E** $\frac{9}{4}$.

Solution: Since $\log_{27}(x) = 3$ we have, since $\log_3(x)$ is the inverse function of 3^x, that

$$x = 27^3 = (3^3)^3 = 3^9.$$

As such, using the exponent rule for the logarithm,

$$\log_3(x^4) = 4\log_3(x) = 4\log_3(3^9) = 4 \cdot 9 = 36.$$

2. (**8 Points**) What is the domain of the function $f(x) = \frac{1}{x} + \sqrt{-x^2 + x + 12}$?

★ **A** $[-3, 0) \cup (0, 4]$.
○ **B** $(-3, 4)$.
○ **C** All real numbers.
○ **D** $(-3, 0) \cup (0, 4)$.
○ **E** $[-3, 4]$.

Solution: We have that $\frac{1}{x}$ is not defined when $x = 0$ so 0 cannot be in our domain. Moreover, $\sqrt{-x^2 + x + 12}$ is only defined when $-x + x + 12$ is nonnegative; since $-x^2 + x + 12 = -(x+3)(x-4)$, this occurs when x is in $[-3, 4]$. We note that $f(x)$ is defined so long as both of $\frac{1}{x}$ and $\sqrt{-x^2 + x + 12}$ are. Putting these two facts together, we obtain that our domain is $[-3, 0) \cup (0, 4]$.

Name: ______________________________ **PUID:** ____________________

3. (**8 Points**) Find all θ in the interval $[0, \pi]$ satisfying $\cos(2\theta) = \dfrac{1}{2}$.

   ◯ **A** $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$.

   ◯ **B** $\theta = \frac{\pi}{3}$.

   ◯ **C** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

   ◯ **D** $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$.

   ★ **E** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

Solution: From facts about special triangles we know that $\cos(\theta) = \frac{1}{2}$ when $\theta = \frac{\pi}{3}$, Remembering that $\cos(\theta)$ is positive in the first and fourth quadrants, we also note that $\cos(\theta) = \frac{1}{2}$ when $\theta = \frac{5\pi}{3}$, and that these are the only solutions to $\cos(\theta) = \frac{1}{2}$ when θ is in $[0, 2\pi]$. As such, the solutions to $\cos(2\theta)$ for θ in $[0, \pi]$ are the solutions to θ is in $[0, 2\pi]$ divided by 2, notably they are $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

Name: ______________________________ **PUID:** ____________________

4. (**8 Points**) Let $f(x) = (x+1)^2 + 9$ with domain restricted to $(-\infty, -1]$ and let $g(x)$ be the inverse of $f(x)$. What are the domain and range of $g(x)$?

   ○ **A** The domain is $(-\infty, -1]$ and the range is $[9, \infty)$.

   ○ **B** The domain is all real numbers and the range is $(-\infty, -1]$.

   ○ **C** The domain is $[9, \infty)$ and range is $[-1, \infty)$.

   ○ **D** The domain is $(-\infty, -1]$ and the range is all real numbers.

   ★ **E** **The domain is $[9, \infty)$ and the range is $(-\infty, -1]$.**

Solution: The range of $g(x)$ is the domain of $f(x)$ which is given as $(-\infty, -1]$. The domain of $g(x)$ is the range of $f(x)$. Since $f(x)$ is a parabola with vertex at $(-1, 9)$, when restricted to $(-\infty, -1]$ the range of $f(x)$ is $[9, \infty)$. As such, the domain of $g(x)$ is $[9, \infty)$.

Name: ______________________________ **PUID:** ____________________

5. (**8 Points**) Find all of the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x+2)(x-4)^3}{(2x-3)^3(x+5)^2}.$$

- ◯ **A** Horizontal: $y = 0$; vertical: $x = \frac{3}{2}$.
- ★ **B** **Horizontal:** $y = 0$**; vertical:** $x = \frac{3}{2}$ **and** $x = -5$**.**
- ◯ **C** Horizontal: $y = 0$; vertical: $x = -2$ and $x = 4$.
- ◯ **D** Horizontal: $y = \frac{1}{2}$; vertical: $x = \frac{3}{2}$ and $x = -5$.
- ◯ **E** Horizontal: $y = \frac{1}{2}$; vertical: $x = -2$ and $x = 4$.

Solution: Since $f(x)$ is a rational function and the degree of the numerator of $f(x)$ is smaller than that of the denominator, we have that

$$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$$

so that $f(x)$ has the horizontal asymptote $y = 0$. Moreover, since $f(x)$ has powers of the terms $2x-3$ and $x+5$ in its denominator, its denominator consists solely of powers of those terms, and that those terms are not cancelled by terms in the numerator, $f(x)$ has $x = \frac{3}{2}$ and $x = -5$ as vertical asymptotes.

Name: ______________________________ **PUID:** ____________________

6. (**8 Points**) Compute

$$\lim_{x\to 3^-} \frac{1-x^2}{x-3}.$$

- ◯ **A** $-\infty$.
- ★ **B** ∞.
- ◯ **C** 6.
- ◯ **D** -6.
- ◯ **E** 0.

Solution: For x smaller than but close to 3, for example for x in $(2,3)$, we have that $1-x^2$ and $x-3$ are both negative so that the quantity $\frac{1-x^2}{x-3}$ is positive for x in $(2,3)$. Moreover, since $\lim_{x\to 3} 1-x^2 = -8 \neq 0$ and $\lim_{x\to 3} x-3 = 0$, the function $f(x) = \frac{1-x^2}{x-3}$ must have a vertical asymptote at $x = 3$. Since $\frac{1-x^2}{x-3}$ is positive for x in $(2,3)$ we must therefore have that

$$\lim_{x\to 3^-} \frac{1-x^2}{x-3} = \infty.$$

Name: ______________________________ **PUID:** ____________________

7. (**8 Points**) Compute

$$\lim_{x\to 0^+} x^{x+2}.$$

Hint: It may help to use the fact that

$$\lim_{x\to 0^+} x\ln(x) = 0.$$

- ◯ **A** 2.
- ◯ **B** $-\infty$.
- ◯ **C** 1.
- ★ **D** 0.
- ◯ **E** ∞.

Solution: We saw in class that $\lim_{x\to 0^+} x^x = 0$ since by composition of limits we have

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln(x)} = e^{\lim_{x\to 0^+} x\ln(x)} = e^0 = 1.$$

As such, we have by the product of limits law that

$$\lim_{x\to 0^+} x^{x+2} = \lim_{x\to 0^+} x^x \lim_{x\to 0^+} x^2 = 1\cdot 0 = 0.$$

Name: ______________________________ **PUID:** ____________________

8. (**8 Points**) Compute

$$\lim_{x \to 9^+} \frac{\sqrt{x} - 3}{x - 5\sqrt{x} + 6}.$$

- ◯ **A** 0.
- ★ **B** 1.
- ◯ **C** ∞.
- ◯ **D** $-\infty$.
- ◯ **E** $\frac{1}{7}$.

Solution: We note that

$$\frac{\sqrt{x} - 3}{x - 5\sqrt{x} + 6} = \frac{\sqrt{x} - 3}{(\sqrt{x} - 3)(\sqrt{x} - 2)}$$

and hence, since when computing a limit as $x \to 9^+$ we may assume $x \neq 9$, we may divide by the non-zero quantity $\sqrt{x} - 3$. We therefore have

$$\lim_{x \to 9^+} \frac{\sqrt{x} - 3}{x - 5\sqrt{x} + 6} = \lim_{x \to 9^+} \frac{\sqrt{x} - 3}{(\sqrt{x} - 3)(\sqrt{x} - 2)} = \lim_{x \to 9^+} \frac{1}{\sqrt{x} - 2} = \frac{1}{3 - 2} = 1.$$

Name: ______________________________ **PUID:** ____________________

9. Let

$$f(x) = \frac{1}{x+2}.$$

(i) (**6 Points**) Use the definition of the derivative to compute $f'(-1)$. To obtain full credit you must show and justify all of your work.

Solution: We use the definition and compute

$$\begin{aligned} f'(-1) &= \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} \\ &= \lim_{h\to 0} \frac{1-(1+h)}{h(1+h)} \\ &= \lim_{h\to 0} \frac{-h}{h(1+h)} \\ &= \lim_{h\to 0} \frac{-1}{1+h} \\ &= -1 \end{aligned}$$

Name: ____________________________ **PUID:** ____________________

(*Note: this is a continuation of Problem 9.*)

(ii) (**6 Points**) Find real numbers a and b such that $y = ax + b$ is the equation for the tangent line to $y = \dfrac{1}{x+2}$ at $x = -1$.

a= -1 b= 0

Solution: We use the slope-point formula to find a line with slope -1 which passes through the point $(-1, 1)$: we solve

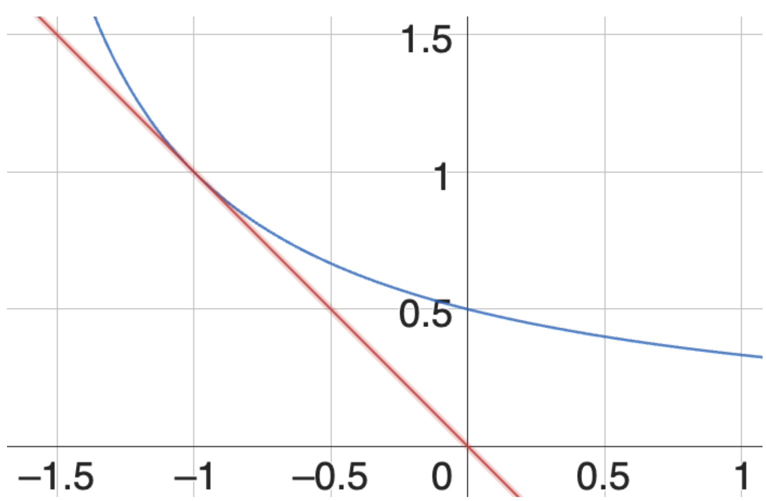
$$\frac{y-1}{x+1} = -1$$

from which we obtain

$$y = -x$$

which is to say that $a = -1$ and $b = 0$.

(iii) (**4 Points**) Draw the tangent line to $y = \dfrac{1}{x+2}$ at $x = -1$ on the following graph.



Name: ______________________________ **PUID:** ____________________

10. Let $g(x)$ be the piecewise function defined as follows:

$$g(x) = \begin{cases} x+3 & \text{if } x \leq -1 \\ (x-1)^2 - 2 & \text{if } -1 < x < 2 \\ -x+3 & \text{if } 2 \leq x < 5 \\ x-6 & \text{if } 5 \leq x \end{cases}$$

(i) (**3 Points**) Determine whether or not $g(x)$ is continuous at $x = -1$. Indicate your answer by filling in the appropriate circle. Show your work.

The function is: ★ **continuous at** $x = -1$**.** ◯ not continuous at $x = -1$.

Solution: We compute

$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} x + 3 = 2$$

and

$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} (x-1)^2 - 2 = 2$$

and thus we have that

$$\lim_{x \to -1} g(x) = 2.$$

We further note that $g(-1) = -1 + 3 = 2$. Since $g(-1) = \lim_{x \to -1} g(x)$ we have that $g(x)$ is continuous at $x = -1$.

Name: ______________________________ **PUID:** ____________________

(*Note: this is a continuation of Problem 10.*)

(ii) (**3 Points**) Determine whether or not $g(x)$ is continuous at $x = 2$. Indicate your answer by filling in the appropriate circle. Show your work.

The function is: ◯ continuous at $x = 2$. ★ **not continuous at** $x = 2$**.**

Solution: We compute

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (x-1)^2 - 2 = -1$$

and

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} -x + 3 = 1$$

and thus we have that $\lim_{x\to 2} g(x)$ does not exist. As such, $g(x)$ is not continuous at $x = 2$.

(iii) (**3 Points**) Determine whether or not $g(x)$ is continuous at $x = 5$. Indicate your answer by filling in the appropriate circle. Show your work.

The function is: ◯ continuous at $x = 5$. ★ **not continuous at** $x = 5$**.**

Solution: We compute

$$\lim_{x\to 5^-} g(x) = \lim_{x\to 5^-} -x + 3 = -2$$

and

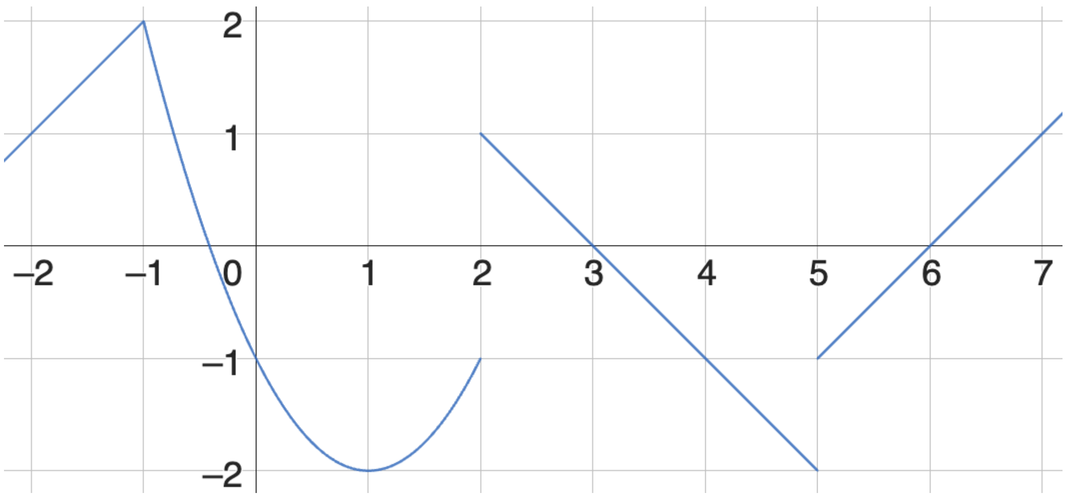
$$\lim_{x\to 5^+} g(x) = \lim_{x\to 5^+} x - 6 = -1$$

and thus we have that $\lim_{x\to 5} g(x)$ does not exist. As such, $g(x)$ is not continuous at $x = 5$.

Name: ______________________________ **PUID:** ____________________

(*Note: this is a continuation of Problem 10.*)

(iv) (**3 Points**) Sketch the graph of $y = g(x)$ on the following axes.



(v) (**4 Points**) Using your graph above, sketch the graph of $y = g'(x)$ on the following axes.