Name: ______________________________ **PUID:** ________________________

Your TA's Name: _________________________ **Recitation Time:** ______________

MA 16100

SPRING 2024

EXAM 2 VERSION A

The problems are numbered 1-10.

For problems 1-8 indicate your answer by filling in the appropriate circle next to the answer choice. Partial credit will **not** be awarded for problems 1-8.

This exam is out of 100 points. Problems 1-8 are worth 8 points each and problems 9 and 10 are worth 16 points each. You will receive 4 points for signing the bottom of this page.

Extra scratch paper is not permitted. Write all your work in this exam booklet.

Write your name and PUID on each page. This will help us locate and successfully grade your test if the pages become separated.

You may not leave the room before 20 minutes have passed. If you finish the exam between when 20 and 50 minutes have passed, you may leave the room after turning in the exam booklet. If you finish within the last 10 minutes of the exam, you MUST REMAIN SEATED until your TA comes and collects your exam booklet.

Exam Policies:

1. Students may not open the exam until instructed to do so.
2. Students must obey the orders and requests by all proctors, TAs, and the instructor.
3. No student may leave in the first 20 min or in the last 10 min of the exam.
4. Books, notes, calculators, or any electronic devices are not allowed on the exam, and they should not even be in sight in the exam room. Students may not look at anybody else's test, and may not communicate with anybody else except, if they have a question, with their TA or the instructor.
5. After time is called, the students have to put down all writing instruments and remain in their seats, while the TAs will collect the exams.
6. Any violation of these rules and any act of academic dishonesty may result in severe penalties. Additionally, all violators will be reported to the Office of the Dean of Students.

I have read and understand the exam rules stated above:

Student Signature: ____

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1. (**8 Points**) What is the derivative of $f(x) = x^e + \log_{17}(x)$?

○ **A** $f'(x) = x^e + \dfrac{\ln(17)}{x}$.

√ **B** $f'(x) = ex^{e-1} + \dfrac{1}{\ln(17)x}$.

○ **C** $f'(x) = x^e + \dfrac{1}{\ln(17)x}$.

○ **D** $f'(x) = ex^{e-1} + \dfrac{\ln(17)}{x}$

○ **E** $f'(x) = ex^{e-1} + \dfrac{\log_{17}(x)}{\ln(17)}$.

This follows from the power rule and the differentiation rule for logarithmic functions.

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2. (**8 Points**) What is the derivative of $f(x) = \dfrac{\sin(x)}{e^x + x^2}$?

√ **A** $\dfrac{e^x \cos(x) - e^x \sin(x) + x^2 \cos(x) - 2x \sin(x)}{(e^x + x^2)^2}$.

○ **B** $\dfrac{-e^x \cos(x) + e^x \sin(x) - x^2 \cos(x) + 2x \sin(x)}{(e^x + x^2)^2}$.

○ **C** $\dfrac{\cos(x)}{e^x + 2x}$.

○ **D** $\dfrac{e^x \cos(x) + e^x \sin(x) + x^2 \cos(x) + 2x \sin(x)}{(e^x + x^2)^2}$.

○ **E** $\dfrac{-e^x - 2x}{(e^x + x^2)^2}$.

This follows from the quotient rule, the power rule, and that $\frac{d}{dx} \sin(x) = \cos(x)$ and $\frac{d}{dx} e^x = e^x$.

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3. (**8 Points**) What is the derivative of $f(x) = x^3 \cos(x) e^x$.

- ◯ **A** $-3x^2 \sin(x) e^x$.
- ◯ **B** $3x^2 \cos(x) e^x - x^3 \sin(x) e^x$.
- ◯ **C** $\left((x^3 + 3x^2)\cos(x) + x^3 \sin(x)\right) e^x$.
- ◯ **D** $x^3 \cos(x) e^x$.
- $\surd$ **E** $\left((x^3 + 3x^2)\cos(x) - x^3 \sin(x)\right) e^x$.

This follows from two uses of the product rule.

$$
\begin{aligned}
\frac{d}{dx}(x^3 \cos(x) e^x) &= x^3 \frac{d}{dx}(\cos(x) e^x) + 3x^2 \cos(x) e^x \\
&= x^3(\cos(x) e^x - \sin(x) e^x) + 3x^2 \cos(x) e^x \\
&= \left((x^3 + 3x^2)\cos(x) - x^3 \sin(x)\right) e^x
\end{aligned}
$$

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4. (**8 Points**) What is the derivative of the function $f(x) = \arctan\left(\frac{1}{x^2}\right)$?

○ **A** $f'(x) = \dfrac{-2}{x\sqrt{x^4-1}}$.

○ **B** $f'(x) = \dfrac{-2}{x\sqrt{x^4+1}}$.

○ **C** $f'(x) = \dfrac{-2}{x^3+x}$.

√ **D** $f'(x) = \dfrac{-2x}{x^4+1}$.

○ **E** $f'(x) = \dfrac{-2}{x\sqrt{x^4-x^2}}$.

This follows from the chain rule, the power rule, and that $\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$.

$$\begin{aligned}\frac{d}{dx}\arctan\left(\frac{1}{x^2}\right) =& \frac{1}{\left(\frac{1}{x^2}\right)^2+1}\frac{-2}{x^3} \\ =& \frac{-2}{x^{-1}+x^3} \\ =& \frac{-2x}{x^4+1}\end{aligned}$$

5. (**8 Points**) A ball is thrown upwards on the surface of the moon and its position above the moon's surface given in metres by $p(t) = -0.8t^2 + 16t + 2$ where the time t is measured in seconds. What is the maximum height in metres that the ball achieves and after how many seconds does it achieve this height?

   ◯ **A** It reaches its maximum height of $78m$ after 10 seconds.

   ◯ **B** It reaches its maximum height of $81.28m$ after 12 seconds.

   ◯ **C** It reaches its maximum height of $82m$ after 8 seconds.

   √ **D It reaches its maximum height of $82m$ after 10 seconds.**

   ◯ **E** It reaches its maximum height of $78m$ after 8 seconds.

We can either solve for the vertex of the parabola or compute the derivative to find the maximum and when it occurs; let's do the former. We have

$$-0.8t^2 + 16t + 2 = -0.8(t^2 - 20t) + 2 = -0.8((t - 10)^2 - 100) + 2 = -0.8(t - 10)^2 + 82$$

and hence the maximum is 82 metres and it occurs at $t = 10$ seconds.

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6. (**8 Points**) Find $\frac{dy}{dx}$ for the relation

$$\tan(y) + 2^y = x.$$

- ○ **A** $\dfrac{dy}{dx} = \dfrac{x}{\sec^2(y) + \frac{1}{\ln(2)}2^y}$.
- ○ **B** $\dfrac{dy}{dx} = \sec(y)\tan(y) + \ln(2)2^y$.
- ○ **C** $\dfrac{dy}{dx} = \dfrac{1}{\sec(y)\tan(y) + \ln(2)2^y}$.
- ○ **D** $\dfrac{dy}{dx} = \dfrac{1}{\sec^2(y) + \frac{1}{\ln(2)}2^y}$.
- √ **E** $\dfrac{dy}{dx} = \dfrac{1}{\sec^2(y) + \ln(2)2^y}$.

We use implicit differentiation, the chain rule, and the facts that $\frac{d}{dx}\tan(x) = \sec^2(x)$ and $\frac{d}{dx}2^x = \ln(2)2^x$ and thus we obtain

$$1 = \frac{d}{dx}x = \frac{d}{dx}(\tan(y) + 2^y) = \sec^2(y)\frac{dy}{dx} + \ln(2)2^y\frac{dy}{dx} = \frac{dy}{dx}\left(\sec^2(y) + \ln(2)2^y\right)$$

and hence that

$$\frac{dy}{dx} = \frac{1}{\sec^2(y) + \ln(2)2^y}$$

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7. (**8 Points**) A 10 foot ladder is leaning against a vertical wall when Charlotte begins pulling the foot of the ladder away from the wall at a rate of 2 feet per second. Assuming the foot of the ladder never leaves the ground and that the top of the ladder always touches the wall, how fast is the top of the ladder sliding down the wall when the foot of the ladder is 6 feet from the wall?

   ◯ **A** 2 feet per second downwards.

   √ **B** 1.5 **feet per second downwards.**

   ◯ **C** 0.75 feet per second downwards.

   ◯ **D** 0.375 feet per second downwards.

   ◯ **E** 0.125 feet per second downwards.

Let's let h denote the height of the ladder on the wall and ℓ denote the distance from the wall to the foot of the ladder. We are given that $\frac{d\ell}{dt} = 2$ and by the chain rule have

$$\frac{dh}{dt} = \frac{dh}{d\ell}\frac{d\ell}{dt} = 2\frac{dh}{d\ell}$$

and moreover since $h^2 + \ell^2 = 100$ so that $h = \sqrt{100 - \ell^2}$ we have that $\frac{dh}{d\ell} = \frac{-\ell}{\sqrt{100-\ell^2}}$; putting all of this together we get at $\ell = 6$ that

$$\frac{dh}{dt} = 2\frac{-6}{\sqrt{100 - 36}} = -\frac{12}{8} = -\frac{3}{2}.$$

Since this value is negative the height of the ladder is decreasing and hence is moving downward, and it is moving at 1.5 feet per second.

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8. (**8 Points**) What are the maximum and minimum values of the function $f(x) = x^4 - 8x^2$ on the interval $[-1, 3]$.

   $\surd$ **A The maximum value is 9 and the minimum value is -16.**

   ◯ **B** The maximum value is 0 and the minimum value is -16.

   ◯ **C** The maximum value is 9 and the minimum value is -7.

   ◯ **D** The maximum value is 0 and the minimum value is -7.

   ◯ **E** The maximum value is 128 and the minimum value is -16.

We compute $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$ so the critical points within $(-1, 3)$ are 0 and 2. We compute the function values at the critical points and the end points: $f(-1) = -7$, $f(0) = 0$, $f(2) = 16 - 32 = -16$, $f(3) = 81 - 72 = 9$. Since the maximum and minimum must be among these values, the maximum value is 9 and the minimum value is -16.

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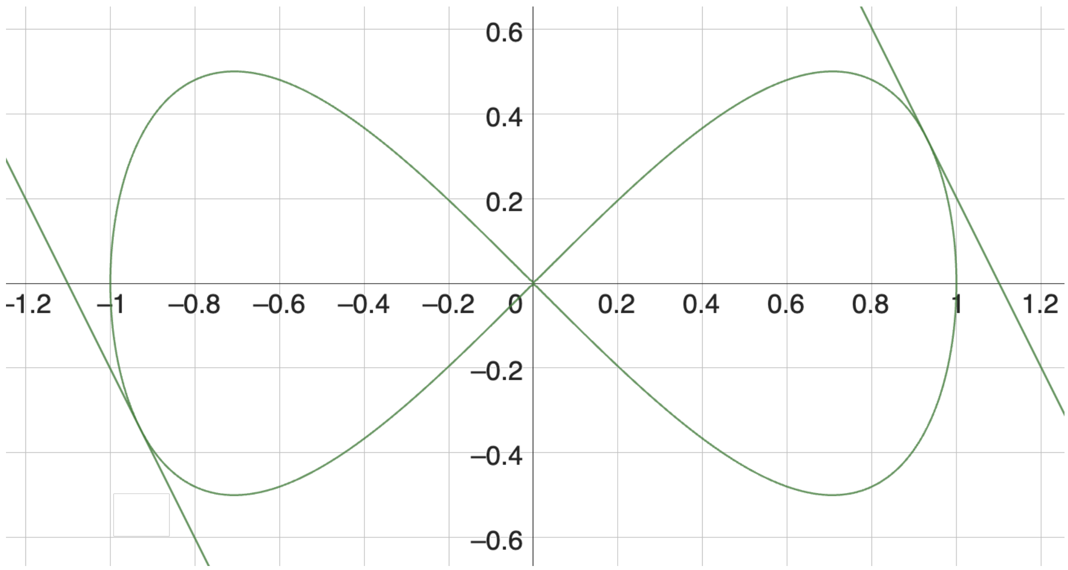
9. An example of a curve called a Lemniscate of Gerono is given by the equation

$$x^4 - x^2 + y^2 = 0.$$

Here is the graph of this curve:



(i) (**6 Points**) On the graph above, **without** doing any computations, draw the tangent lines at the two points on the curve $x^4 - x^2 + y^2 = 0$ where $\dfrac{dy}{dx} = -2$.

(*Note*: Your line drawing does not have to be precise! It just needs to give some indication of where these points occur.)

The correct tangent lines are indicated above.

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(*Note: this is a continuation of Problem 9.*)

(ii) (**6 Points**) Compute $\dfrac{dy}{dx}$ for the curve $x^4 - x^2 + y^2 = 0$.

We implicitly differentiate and obtain

$$4x^3 - 2x + 2y\frac{dy}{dx} = 0$$

which can be rearranged to

$$\frac{dy}{dx} = \frac{x - 2x^3}{y}$$

(iii) (**4 Points**) Find all points on the curve $x^4 - x^2 + y^2 = 0$ where $\dfrac{dy}{dx} = 0$.

We solve

$$0 = \frac{dy}{dx} = \frac{x - 2x^3}{y}$$

so we much solve $x - 2x^3 = 0$ which has solutions $x = 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$. If $x = 0$ we solve

$$0 = 0^4 - 0^2 + y^2 = y^2$$

which has the solution $y = 0$; the value $\frac{dy}{dx}$ is undefined as $(0, 0)$ so it is NOT one of our desired points.

For $x = \pm\frac{1}{\sqrt{2}}$ we solve

$$0 = \left(\pm\frac{1}{\sqrt{2}}\right)^4 - \left(\pm\frac{1}{\sqrt{2}}\right)^2 + y^2 = \frac{1}{4} - \frac{1}{2} + y^2$$

so that $y^2 = \frac{1}{4}$ and hence that $y = \pm\frac{1}{2}$.

As such, we obtain the four points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right)$ as precisely those where $\frac{dy}{dx} = 0$.

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10. An empty spherical balloon is being blown up at a rate of $\frac{36\pi}{5}$ cubic metres per second. Recall that if r is the radius of a sphere, the volume of that sphere is $V = \frac{4\pi}{3}r^3$ and the surface area of that sphere is $A = 4\pi r^2$.

    (i) (**3 Points**) Compute $\frac{dV}{dr}$. Show your work and write your answer on the line below.

    $$\frac{dV}{dr} = 4\pi r^2$$

    Here we use the power rule.

    (ii) (**3 Points**) Compute $\frac{dA}{dr}$. Show your work and write your answer on the line below.

    $$\frac{dA}{dr} = 8\pi r.$$

    Again we use the power rule.

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(*Note: this is a continuation of Problem 10.*)

(iii) (**2 Points**) What is the radius of the balloon after 5 seconds? Show your work and write your answer on the line below.

The balloon's radius is 3 **metres.**

Since the balloon was empty to start with, after 5 seconds the volume of the balloon is $5\frac{dV}{dt} = 5\frac{36\pi}{5} = 36\pi$. We solve for the radius with the volume formula

$$36\pi = V = \frac{4}{3}\pi r^3$$

so that $r^3 = 27$ and hence that $r = 3$.

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(*Note: this is a continuation of Problem 10.*)

(iv) (**4 Points**) How fast is the radius of the balloon changing after 5 seconds? Show your work and write your answer on the line below.

The balloon's radius is changing at a rate of $\frac{1}{5}$ metres per second.

We have that

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

and we know $\frac{dV}{dt} = \frac{36\pi}{5}$, that $r = 3$ at $t = 5$ seconds and that $\frac{dV}{dr} = 4\pi r^2$ and hence that $\frac{dV}{dr} = 36\pi$ at $r = 3$. Putting all of this together we obtain

$$\frac{36\pi}{5} = 36\pi\frac{dr}{dt}$$

so that $\frac{dr}{dt} = \frac{1}{5}$ at $t = 5$ seconds.

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(*Note: this is a continuation of Problem 10.*)

(v) (**4 Points**) How fast is the surface area of the balloon changing after 5 seconds? Show your work and write your answer on the line below.

The balloon's surface area is changing at a rate of $\frac{24\pi}{5}$ metres squared per second.

We have that

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}.$$

At $t = 5$ seconds we have $r = 3$ and $\frac{dA}{dr} = 8\pi r$ so that $\frac{dA}{dr} = 24\pi$ at $t = 5$ seconds. Moreover, we know from the previous part that $\frac{dr}{dt} = \frac{1}{5}$ at $t = 5$ seconds and therefore at $t = 5$ seconds we have

$$\frac{dA}{dt} = \frac{24\pi}{5}$$

(*This page may be used for scratch work but work done here will not be graded.*)