Name: ______________________________ **PUID:** ________________________

Your TA's Name: _________________________ **Recitation Time:** ______________

MA 16100

SPRING 2024

EXAM 3 VERSION A

The problems are numbered 1-10.

For problems 1-8 indicate your answer by filling in the appropriate circle next to the answer choice. Partial credit will **not** be awarded for problems 1-8.

This exam is out of 100 points. Problems 1-8 are worth 8 points each and problems 9 and 10 are worth 16 points each. You will receive 4 points for signing the bottom of this page.

Extra scratch paper is not permitted. Write all your work in this exam booklet.

Write your name and PUID on each page. This will help us locate and successfully grade your test if the pages become separated.

You may not leave the room before 20 minutes have passed. If you finish the exam between when 20 and 50 minutes have passed, you may leave the room after turning in the exam booklet. If you finish within the last 10 minutes of the exam, you MUST REMAIN SEATED until your TA comes and collects your exam booklet.

Exam Policies:

1. Students may not open the exam until instructed to do so.
2. Students must obey the orders and requests by all proctors, TAs, and the instructor.
3. No student may leave in the first 20 min or in the last 10 min of the exam.
4. Books, notes, calculators, or any electronic devices are not allowed on the exam, and they should not even be in sight in the exam room. Students may not look at anybody else's test, and may not communicate with anybody else except, if they have a question, with their TA or the instructor.
5. After time is called, the students have to put down all writing instruments and remain in their seats, while the TAs will collect the exams.
6. Any violation of these rules and any act of academic dishonesty may result in severe penalties. Additionally, all violators will be reported to the Office of the Dean of Students.

I have read and understand the exam rules stated above:

Student Signature: ____

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1. (**8 Points**) On which of the following intervals is function $f(x) = x^3 + x^2 - x$ both decreasing and concave up?

   √ **A** $\left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$.

   ○ **B** $\left(-1, -\dfrac{1}{3}\right)$.

   ○ **C** $\left(\dfrac{1}{3}, \infty\right)$.

   ○ **D** $(-\infty, -1)$.

   ○ **E** $\left(-\dfrac{1}{3}, \infty\right)$.

Solution: We have that

$$f'(x) = 3x^2 + 2x - 1 = (3x-1)(x+1)$$

which is positive on $(-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$ and is negative on $\left(-1, \frac{1}{3}\right)$ and we have

$$f''(x) = 6x + 2$$

which is positive on $\left(-\frac{1}{3}, \infty\right)$ and negative on $\left(-\infty, -\frac{1}{3}\right)$. For $f(x)$ to be decreasing and concave up we need $f'(x) < 0$ and $f''(x) > 0$; this occurs on the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

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2. (**8 Points**) Let $f(x)$ be a function continuous on $[2,7]$ and differentiable on $(2,7)$ where we have that $f(2)=1$, $f(4)=7$, $f(5)=5$, $f(7)=1$. Which of the following is **not necessarily** a value of $f'(x)$ for some x in $(2,7)$?

   ◯ **A** 0.

   √ **B** 6.

   ◯ **C** 3.

   ◯ **D** $\frac{4}{3}$.

   ◯ **E** -2.

Solution: By the Mean Value Theorem, $f'(x)$ must take on the value 0 for some c in $(2,7)$ since $\frac{f(7)-f(2)}{7-5}=0$; it similarly must take on the value 3 for some c in $(2,4)$, the value $\frac{4}{3}$ for some c in $(2,5)$ and the value -2 for some c in $(4,5)$, each due to applications of the Mean Value Theorem. The only value which we don't know for sure will necessarily appear as a value for $f'(x)$ is 6.

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3. (**8 Points**) How many critical points does the function $f(x) = |x^3 - 3x|$ have?

   ◯ **A** 1.

   ◯ **B** 2.

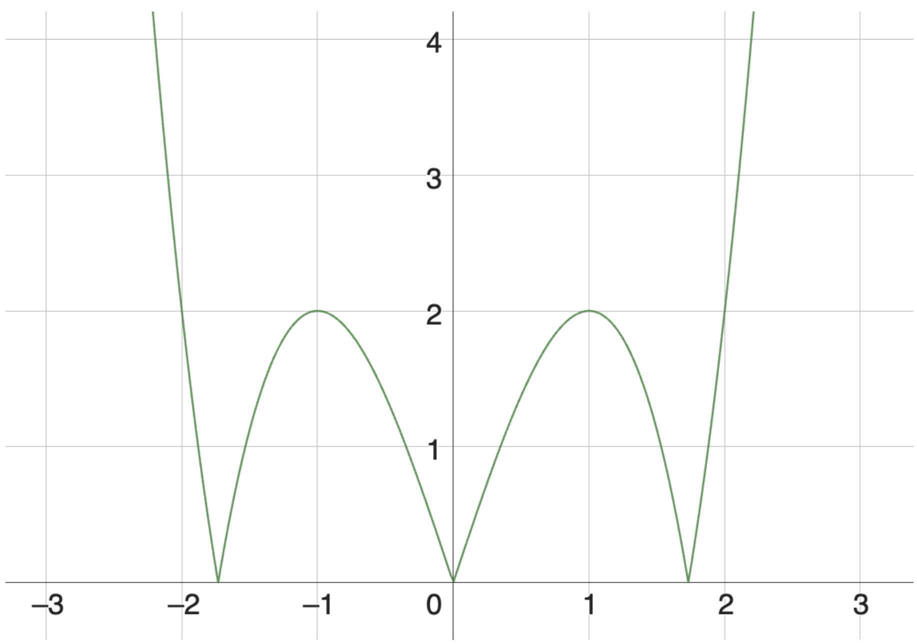
   ◯ **C** 3.

   ◯ **D** 4.

   √ **E** 5.

Solution: Let $g(x) = x^3 - 3x$. The function $f(x)$ is not differentiable at any point x where $g(x) = 0$ and $g'(x) \neq 0$ since taking the absolute value in this case will cause the graph of $f(x)$ to have a "sharp point" at which it is not differentiable. This occurs at the three points $x = 0, \pm\sqrt{3}$. We also see that $f'(x) = 0$ at $x = \pm 1$. All together, $f(x)$ has five critical points.

A graph of $y = f(x)$ is pictured below.



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4. (**8 Points**) Let $f(x)$ be differentiable for all real numbers where

$$f'(x) = (x+4)^2(x-1)(x-3)^3(x-5).$$

How many local maxima does $f(x)$ have?

- ◯ **A** 0.
- √ **B** 1.
- ◯ **C** 2.
- ◯ **D** 3.
- ◯ **E** 4.

Solution: We see that $f'(x)$ is negative for $x < -4$, negative for $-4 < x < 1$, positive for $1 < x < 3$, negative for $3 < x < 5$, and positive for $5 < x$. By the First Derivative Test, the point $x = 3$ is a local maximum for $f(x)$ and no other critical points for $f(x)$ are local maxima.

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5. (**8 Points**) How many local minima does the function $f(x) = x^4 - \frac{4}{3}x^3 - 12x^2$ have?

   ◯ **A** 0.

   ◯ **B** 1.

   √ **C** 2.

   ◯ **D** 3.

   ◯ **E** 4.

Solution: We compute

$$f'(x) = 4x^3 - 4x^2 - 24x = 4x(x^2 - x - 6) = 4x(x+2)(x-3)$$

so that $f(x)$ has critical points at $x = -2, 0, 3$. Either of the First Derivative Test or the Second Derivative Test shows that $x = -2$ and $x = 3$ are local minima and $x = 0$ is a local maximum.

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6. (**8 Points**) A farmer needs to plant a rectangular field directly along a river. To protect the field, they need to build a fence around three of its sides (the fourth side along the river does not require a fence). Suppose the farmer has materials to build 1200m of fencing. If ℓ denotes the length of the side of the fence parallel to the river and w denotes the length of the sides of the fence perpendicular to the river, what dimensions of the field will maximize its area?

 ◯ **A** $\ell = 200, w = 500$.

 ◯ **B** $\ell = 400, w = 400$.

 √ **C** $\ell = 600, w = 300$.

 ◯ **D** $\ell = 800, w = 200$.

 ◯ **E** $\ell = 1000, w = 100$.

Solution: We have that $\ell + 2w = 1200$ is the given information and our desired information is the maximum area where area is given by $A = \ell w$. The given information allows us to note that $\ell = 1200 - 2w$ and hence that $A = A(w) = 1200w - 2w^2$ as a function of w. We solve $A'(w) = 1200 - 4w$ and hence we have a critical point at $w = 300$. We now note that both ℓ and w must be positive and hence we require that w is in $[0, 600]$; we moreover see that $A(0) = 0 = A(600)$. As such, the maximum occurs at $w = 300$ since $A(300) = 18,000$ which is greater than 0. As such, we must have $w = 300$ and $\ell = 1200 - 2(300) = 600$.

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7. (**8 Points**) Which of the following is the best linear approximation to $y = \sqrt{x}$ at the point $x = 16$.

   ○ **A** $y = \frac{1}{2}x + 2$

   ○ **B** $y = \frac{1}{4}x$

   ○ **C** $y = \frac{1}{8}x + 4$

   ○ **D** $y = \frac{1}{2}x - 4$

   √ **E** $y = \frac{1}{8}x + 2$

Solution: We have $f'(x) = \dfrac{1}{2\sqrt{x}}$ so that $f'(16) = \dfrac{1}{8}$. Thus the equation for the tangent line $f(x)$ is

$$L(x) = f(16) + f'(16)(x - 16) = 4 + \frac{1}{8}(x - 16) = \frac{1}{8}x + 2.$$

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8. (**8 Points**) Compute $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x$.

- ◯ **A** e^3.
- ◯ **B** ∞.
- ◯ **C** 0.
- √ **D** e^{-3}**.**
- ◯ **E** 1.

Solution: We have

$$\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1-\frac{3}{x}\right)} = e^{\lim_{x\to\infty} x\ln\left(1-\frac{3}{x}\right)}$$

so long as the limit $\lim_{x\to\infty} x\ln\left(1-\frac{3}{x}\right)$ exists. This limit is a "$0\cdot\infty$" indeterminate form which may solved using L'Hopital's Rule; we have

$$\lim_{x\to\infty} x\ln\left(1-\frac{3}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1-\frac{3}{x}}\cdot -3\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = -3\lim_{x\to\infty}\frac{1}{1-\frac{3}{x}} = -3$$

and therefore

$$\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x = e^{-3}.$$

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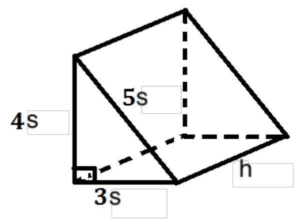
9. A company wants create a closed cardboard box which is shaped like a triangular prism with a right triangle whose sides are in ratio 3 to 4 to 5 as its base and top. Let h denote the height of the box and let s be such that the length of the sides of the triangular base and top of the box are $3s$, $4s$ and $5s$, respectively. The diagram below gives a bottom view of the box.



(i) (**3 Points**)
Write down a formula for the volume of the box in terms of h and s.

$V = 6s^2h$

(ii) (**3 Points**)
Write down a formula for the surface area of the box in terms of h and s.

$A = 12s^2 + 12sh$

(iii) (**4 Points**) The company wishes to build the box using 12 square metres of cardboard. Given this restriction, write a formula for the volume of the box in terms of s. Show all of your work.

$V(s) = 6s - 6s^3$

Solution: Since $12 = A = 12s^2 + 12sh$ we have $s^2 + sh = 1$ or $h = \frac{1}{s} - s$ and thus that

$$A = 6s^2h = 6s^2\left(\frac{1}{s} - s\right) = 6s - 6s^3.$$

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(*Note: this is a continuation of Problem 9.*)

(iv) (**6 Points**) If the box is to be built with 12 square metres of cardboard, find the value of s which maximizes the box's internal volume. Show all of your work.

$s = \frac{1}{\sqrt{3}}$

Solution: We have $V(s) = 6s - 6s^3$ which must be positive and we require s itself to be positive, so we need only care about s in $[0, 1]$ and we see that $V(0) = 0 = V(1)$. We compute $V'(s) = 6 - 18s^2$ which has the positive solution of $s = \frac{1}{\sqrt{3}}$; since

$$V\left(\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} - \frac{6}{3\sqrt{3}} = \frac{4}{\sqrt{3}} > 0$$

we have that $s = \frac{1}{\sqrt{3}}$ must be our desired maximum.

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10. Let $f(x) = \dfrac{1}{x+1} + \dfrac{1}{x-1}$.

(i) (**2 Points**) Determine if $f(x)$ has any vertical asymptotes (write NONE if it has none). Show your work.

The vertical asymptotes are $x = 1$ and $x = -1$.

(ii) (**2 Points**) Determine if $f(x)$ has any horizontal asymptotes (write NONE if it has none). Show your work.

The horizontal asymptote is $y = 0$.

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(*Note: this is a continuation of Problem 10.*)

(iii) (**1 Point**) Find all points x at which $f(x) = 0$.

$f(x) = 0$ **for** $x = 0$**.**
(write "NONE" if $f(x) \neq 0$ for any x)

(iv) (**3 Points**) Determine when $f(x)$ is increasing and decreasing.

$f(x)$ **is increasing on N/A**
(write "N/A" if the function does not increase on any interval)

$f(x)$ **is decreasing on** $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$**.**
(write "N/A" if the function does not decrease on any interval)

Solution: We compute

$$f'(x) = -\frac{1}{(x+1)^2} - \frac{1}{(x-1)^2} = -2\frac{x^2+1}{(x^2-1)^2}$$

which is negative at all points at which it is defined (which is all points except $x = -1, 1$).

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(*Note: this is a continuation of Problem 10.*)

(v) (**3 Points**) Determine when $f(x)$ is concave up and concave down.

$f(x)$ **is concave up on** $(-1,0) \cup (1,\infty)$
(write "N/A" if the function is not concave up on any interval)

$f(x)$ **is concave down on** $(-\infty,-1) \cup (0,1)$
(write "N/A" if the function is not concave down on any interval)

Solution: We compute

$$f''(x) = 2\frac{1}{(x+1)^3} + 2\frac{1}{(x-1)^3} = 4x\frac{x^2+3}{(x^2-1)^3}$$

which is positive on $(-1,0) \cup (1,\infty)$ and negative on $(-\infty,-1) \cup (0,1)$. This is because the sign of x^2+3 is always positive, the sign of $4x$ is positive for $x > 0$ and negative for $x < 0$, and the sign of $(x^2-1)^3$ is positive for x in $(-\infty,1) \cup (1,\infty)$ and negative for x in $(-1,1)$. Putting all this together yields the result.

(vi) (**1 Point**) Find all inflection points of $f(x)$.

$f(x)$ **has an inflection point at** $x = 0$
(write "NONE" if $f(x)$ has no inflection points)

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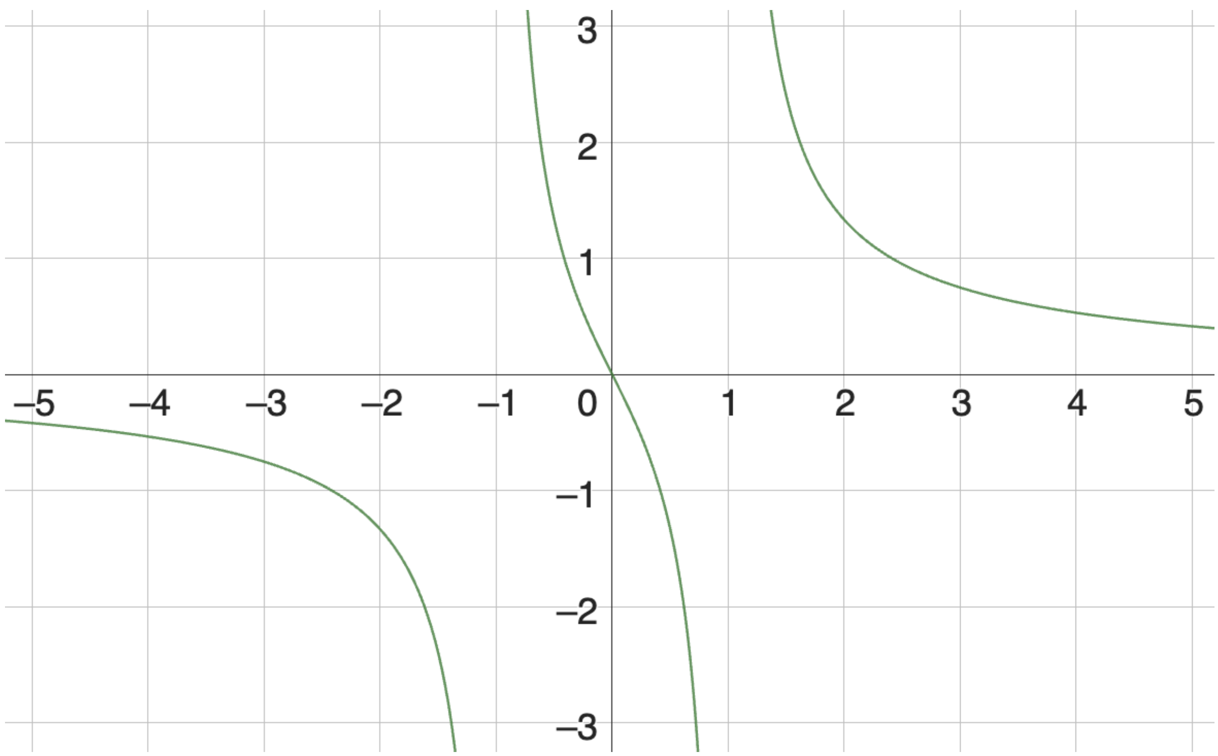
(*Note: this is a continuation of Problem 10.*)

(vii) (**4 Points**) Draw a rough graph of $f(x)$ on the following set of axes indicating the information you determined in parts (i)-(vi).



(*This page may be used for scratch work but work done here will not be graded.*)