1 Cohomology of Projective Schemes

We want to return to the algebraic story.

Fix an algebraically closed field k.

Lemma If $X \subset \mathbb{P}^n_k$ is a closed subscheme all of whose components have $\dim = d$, there exists an open affine cover $\mathcal{U} = \{U_0, \dots U_d\}$

pf We claim we can find hyperplanes $H_0, \dots H_d \subset \mathbb{P}^n_k$, s.t. $X \cap H_0 \cap \dots H_d = \emptyset$

Then we take $U_i := X - H_i$

We can prove the claim by induction on d.

Suppose $d = 1$, i.e. X is a finite collection of (closed) pts.

Then we can find H_0 which avoids these pts, since $k = \bar{k}$.

When $d > 1$, we choose H_d s.t. $X \cap H_d \neq \emptyset$. Since the dimension of $X \cap H_d$, drops we're done by induction //

②

Prop If $X \subset \mathbb{P}^n_k$ is an irreducible closed subscheme, and $\mathcal{F}$ is a coherent sheaf on it.

a) $\dim_k H^i(X, \mathcal{F}) < \infty \quad \forall i$

b) $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$

Pf: Let $i : X \hookrightarrow \mathbb{P}^n_k$ denote the inclusion, then $i_* \mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$ and

$$H^i(X, \mathcal{F}) = H^i(\mathbb{P}^n_k, i_* \mathcal{F})$$

Therefore a) follows from what we proved earlier for $\mathbb{P}^n_k$.

By the previous lemma X has open affine cover

$$\mathcal{U} = \{U_0, \dots U_d\}, \quad d = \dim X.$$

Therefore

$$H^i(X, \mathcal{F}) = \check{H}^i(\mathcal{U}, \mathcal{F}) = 0$$

for $i > d$ //

③

It follows that
$$\dim H^i(X, \mathcal{O}_X), \quad i = 0, .. d$$
gives useful isomorphism invariants

when X is a reduced
$$\dim H^0(X, \mathcal{O}_X) = 1$$
because global regular functions are constant.

In particular when X is a nonsingular irreducible projective curve, we get only one interesting number

Def The genus of X is
$$g = \dim H^1(X, \mathcal{O}_X)$$
We do one computation

Thm If $X \subset \mathbb{P}^2_k$ is defined by a degree d polynomial, $g = \binom{d-1}{2}$

<u>pf</u> Let $i : X \hookrightarrow \mathbb{P}^2_k$ denote the inclusion.

We have an exact seq.

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^2} \to i_* \mathcal{O}_X \to 0$$

If f is a defining polynomial of X,

$$\mathcal{I}_X = \widetilde{(f)} \cong \widetilde{S(-d)} \cong \mathcal{O}_{\mathbb{P}^2}(-d)$$

Also, since i_* is exact,

$$H^1(X, \mathcal{O}_X) \cong H^1(\mathbb{P}^2, i_* \mathcal{O}_X)$$

therefore, we have

$$\underset{\substack{\| \\ 0}}{H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})} \to H^1(X, \mathcal{O}_X) \to H^2(\mathbb{P}^2, \mathcal{O}(-d)) \hookrightarrow \underset{\substack{\| \\ 0}}{H^2(X, \mathcal{O}_X)}$$

Therefore

$$H^1(X, \mathcal{O}_X) \cong H^2(\mathbb{P}^2, \mathcal{O}(-d)) = S_{d-3}$$

where $S = k[x, y, z]$. The dimension is easily computed as stated //

2 Kähler differentials

Let X be a scheme of finite type over an alg closed field k.

Thm / Def There exists a coherent sheaf $\Omega^1_{X/k} = \Omega_{X/k}$ such that $\Omega_{X/k}|_U = \tilde{\Omega}_R$ for any affine open $\operatorname{Spec} R \subset X$. If X/k is a nonsingular variety of dim. n, $\Omega^1_{X/k}$ is locally free of rank n.

See Hartshorne Chap II, sect 8.

3 Serre duality for curves

For the rest of this section

X will denote a nonsingular projective curve (= 1 dim'l scheme) over $k = \bar{k}$. A point of X means a closed point (which is what one means classically by a pt).

Then Ω^1_X is a line bundle, called the canonical line bundle.

Given a divisor D, let

$$\Omega^1_X(D) = \Omega^1_X \otimes \mathcal{O}_X(D)$$

Serre duality (algebraic version)

There is an isomorphism

$$H^1(X, \mathcal{O}(D)) \cong H^0(X, \Omega^1_X(-D))^*$$

for any divisor D on X.

Cor The genus $g = \dim H^0(X, \Omega^1_X)$

Let $k(X)$ = function field of X

We define the ring of adeles by

$$R = \{ (r_p) \in \prod_{p \in x} k(X) \mid r_p \in \mathcal{O}_p \text{ for almost all } p \}$$

when x = closed pts of X

We have a diagonal embedding

$$K(X) \subset R$$
$$f \longmapsto (f)_{P \in X}$$

For each divisor $D = \sum n_P P$

let $R(D) = \{ (r_P) \in R \mid \mathrm{ord}_P r_P \geq -n_P \ \forall P \in \mathrm{supp}\, D \}$.

Prop $\quad H^1(X, \mathcal{O}(D)) = R / (R(D) + K(X))$

pf $\quad$ Let $K(X)_X$ = constant sheaf associated to $K(X)$

We have an exact sequence

$$0 \to \mathcal{O}_X(D) \to K(X)_X \to K(X)_X / \mathcal{O}(D) \to 0$$

Since $K(X)_X$ is flasque, we get

$$\underbrace{H^0(K(X)_X)}_{K(X)} \to H^0(K(X)_X / \mathcal{O}(D)) \to H^1(X, \mathcal{O}_X(D)) \to 0$$

One checks $K(X)_X / \mathcal{O}(D)$ is a sum of skyscraper sheaves $\bigoplus_P K(X)/\mathcal{O}(D)_P \Rightarrow H^0(K(X)_X / \mathcal{O}(D)) = R / R(D)$ //

Set $\Omega = \Omega_{k(X)/k}$. This is should be interepted as the space of rational differential forms on X.

Let $p \in X$. Since $\mathcal{O}_{X,p}$ is a d.v.r with valuation $\mathrm{ord}_p : k(X) \to \mathbb{Z} \cup \{\infty\}$, there exists an element $t \in m_p$ which generates the maximal ideal. t plays the role of a local coordinate, and it is usually called a local uniformizer. It is not unique. The completion of $k(X)$ w.r.t the valuation ord_p is isomorphic to $k((t))$

Thm / Def If $\omega \in \Omega$, the coefficient of $\frac{1}{t} dt$ of the image of ω in $k((t)) dt$ is independent of t, and denoted by $\mathrm{res}_p(\omega)$. The sum

$$\sum_{p \in X} \mathrm{res}_p(\omega) = 0$$

When $k = \mathbb{C}$, these statements follow easily from Cauchy's formula and Stokes' theorem. For the general case see Serre's book "Algebraic Groups and Class fields".

We define a pairing

$$R \times \Omega \longrightarrow k$$

$$\langle (r_p), \omega \rangle = \sum_p \operatorname{res}_p (r_p \omega)$$

Given $\omega \in \Omega$, set

$$\operatorname{ord}_p \omega = \operatorname{ord}_p f, \text{ when } \omega = f\, dt$$

for some local uniformizer

and $(\omega) = \sum_p (\operatorname{ord}_p \omega) p$

Given $D = \sum n_p p$, we can identify

$$H^0(X, \Omega^1_X(-D)) = \{ \omega \in \Omega \mid \operatorname{ord}_p \omega \geq + n_p \}$$

lemma If $(r_p) \in R(D) + k(X)$ and $\omega \in H^0(\Omega(-D))$, then

$$\langle (r_p), \omega \rangle = 0$$

pf If $(r_p) \in R(D)$, $\mathrm{ord}_p\, r_p \geq -n_p$ while $\mathrm{ord}_p\, \omega \geq +n_p$. Therefore

$$\mathrm{res}_p(r_p \omega) = 0$$

If $f \in k(X)$, then $\sum_p \mathrm{res}_p(f\omega) = 0$ by the last thm //

Therefore $\langle\, ,\, \rangle$ induces a pairing

$$\underbrace{R/(R(D) + k(X))}_{H^1(X, \mathcal{O}(D))} \times H^0(\Omega^1_X(-D)) \to k.$$

A precise form of duality is

Thm (Serre) The above pairing is nondegenerate

Full details can be found in Serre's book. However, we can explain the idea. We can write

$$\Omega = \bigcup_D H^0(\Omega(-D)) = \varinjlim_D H^0(\Omega(-D))$$

This is a 1 dim'l vector space over $k(x)$. On the other hand

$$J := \varinjlim_D \left(R / R(D) + k(x) \right)^*$$

can also be given the structure of $k(x)$-v.s.

Prop (a) $\dim_{k(x)} J \leq 1$

(b) $\langle , \rangle$ induces an injective $k(x)$-lin map from $\Omega \to J$

Cor $\Omega \cong J$

To finish the proof, one needs to check the isomorphism is compatible with the filtration induced by D.

4 Riemann-Roch

Let X be a curve = nonsingular reduced irreducible one dim'l projective variety over $k = \bar{k}$.

Let g = genus of X.

Given a divisor $D = \sum n_p p$

Set $\deg D = \sum n_p \in \mathbb{Z}$

and

$$\ell(D) = \{ f \in k(X) \mid \forall p,\ \mathrm{ord}_p f \geq -n_p \} \cup \{0\}$$

We state

Thm (Riemann's inequality)

$$\ell(D) \geq \deg D + 1 - g$$

This will be sharpened to an equality shortly.

Given a nonzero rational 1-form $\omega \in \Omega_{k(X)/k}$, it's divisor

$$(\omega) = \sum_{p \in X} (\mathrm{ord}_p \omega)\, p$$

Lemma If $\omega, \omega' \in \Omega_{k(x)/k} - \{0\}$

$(\omega) \sim (\omega')$.

pf $\omega' = f\omega$ with $f \in k(x)$

and $(\omega') = (\omega) + (f)$ //

Def A canonical divisor is a divisor of the form (ω). This is traditional written as K. (The lemma says that the linear equivalence class is well defined.)

Thm (Riemann-Roch)

$$\ell(D) - \ell(K - D) = \deg D + 1 - g$$