1 Euler characteristics & Riemann-Roch

Given a coherent sheaf $\mathcal{F}$ on a projective scheme Y, we define its Euler characteristic by

$$\chi(Y, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim H^i(Y, \mathcal{F})$$

$$= \sum_{i=0}^{\dim X} (-1)^i \dim H^i(Y, \mathcal{F})$$

Here is the key fact.

Lemma If

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is exact, then

$$\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$$

pf This follows from the long exact sequence and the following additivity property:

If $0 \to V_1 \to V_2 \to \dots \to 0$

is an exact of fin dim vector spaces $\sum (-1)^i \dim V_i = 0$

//

We can now reinterpret and prove Riemann-Roch:

Thm If D is a divisor on a genus g curve

$$\ell(D) - \ell(K-D) = \deg D + 1 - g$$

Before proving, we rewrite the left side as

$$\ell(D) = \dim H^0(X, \mathcal{O}_X(D))$$
$$\ell(K-D) = \dim H^0(X, \mathcal{O}_X(K-D))$$
$$= \dim H^1(X, \Omega_X^1(-D))$$

where we use Serre duality for the last step.

Then the R.R. is equivalent to:

Thm $\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g.$

pf Let $D = \sum n_p p$. We prove this by induction on

$$m(D) = \sum |n_p|$$

The base case is $D = 0$

Then $\chi(X, \mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X)$

$- \dim H^1(X, \mathcal{O}_X)$

$= 1 - g$

by definition.

Next suppose $m(D) > 0$

if $D = n_p p + \cdots$ with $n_p > 0$

set $E = (n_p - 1) p + \cdots$

Then we have an exact seq.

$$0 \to \mathcal{O}(E) \to \mathcal{O}(D) \to k(p) \to 0$$

where $k(p)$ is the skyscraper sheaf at p

Then $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(E)) + \chi(k(p))$

By induction $\chi(\mathcal{O}(E)) = \deg E + 1 - g$

and we also have

$$\chi(k(p)) = \underbrace{\dim H^0(k(p))}_{1} - \underbrace{\dim H^1(k(p))}_{0} = 1$$

So $\chi(\mathcal{O}(D)) = (\deg E + 1) + 1 - g$

If all the nonzero coefficients of D are negative, we can find E with

$$0 \to \mathcal{O}(D) \to \mathcal{O}(E) \to k(p) \to 0$$

and $m(E) < m(D)$ and conclude for similar reasons //

2 Easy application of Riemann-Roch

Let X be a curve of genus g.

Prop $\deg K = 2g-2$

pf: plug $D = K$ into R.R. to obtain

$$\ell(K) - \ell(K-K) = \deg K + 1 - g$$

Given a nonzero rational differential ω, we can take $K = (\omega)$

Then we have an isomorphism

$$\mathcal{O}_X(K) \cong \Omega^1_X$$

defined by

$$\mathcal{O}(D)(U) \longrightarrow \Omega^1_X(U)$$
$$f \longmapsto f \cdot \omega$$

Therefore

$$\ell(K) = \dim H^0(\Omega_X^1) = g$$

Similarly

$$\ell(0) = \dim H^0(\mathcal{O}_X) = 1$$

Therefore

$$g - 1 = \deg K + 1 - g$$

$$\Rightarrow \quad \deg K = 2g - 2$$

//

Given a nonconstant morphism

$$f : X \to Y$$

between curves, we get an extension of fields

$$k(Y) \subseteq k(X)$$

The degree of this extension is called the **degree** of f.

Prop/Def For all $p \in Y$

$$\# f^{-1}(p) \leq \deg f$$

For all but finitely many pts called branch or ramification pts, equality holds.

It's possible to give a more precise statement. If t is uniformizer at p and $q \in f^{-1}(p)$, the ramification index

$$e_q = \mathrm{ord}_q(t) \quad (\text{remember } t \in k(Y) \subseteq k(X))$$

Thm For all $p \in Y$,

$$\sum e_q = \deg f$$

Finally, we have

Thm A degree 1 morphism $f: X \to Y$ is an isomorphism

NB: This is false in higher dimensions or if we allow singularities,

Thm $\mathbb{P}^1_k$ is the only genus 0 curve.

pf By what we proved earlier

$H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}) = 0 \Rightarrow g(\mathbb{P}^1) = 0$

Conversely, suppose $g(X) = 0$

Let $p \in X$ (a closed pt)

By R.R.

$\ell(p) = 1 + 1 - 0 = 2$

Since $H^0(\mathcal{O}(p)) = \{f \in k(X) \mid \mathrm{ord}_p f \geq -1$ & $\mathrm{ord}_q f \geq 0$ $\forall q \neq p\}$

$\supsetneq k$

We must have $f \in H^0(\mathcal{O}(p))$ which is nonconstant.

View $f : X \to \mathbb{P}^1$ as a morphism such that $f^{-1}(\infty) = p$ as a divisor

$\Rightarrow \deg f = 1$ by previous results

$\Rightarrow f$ is is an isomorphism

//

3 Projective Embeddings

A basic problem in classical as well as modern algebraic geometry is to describe maps from a variety or scheme to $\mathbb{P}^n_k$.

The idea is simple, let X be a variety. Given regular functions $f_0, \dots f_n$, set

$F(x) = (f_0(x), \dots f_n(x))$

Then we get a morphism

$$\begin{tikzcd} X \supseteq V = \{x \in X \mid F(x) \neq 0\} \arrow[r] \arrow[dr, "\varphi"'] & \mathbb{A}^{n+1}_k \setminus \{0\} \arrow[d] \\ & \mathbb{P}^n_k \end{tikzcd}$$

There are 2 problems

(1) V might not equal X, so φ is only partially defined

(2) If X is projective, then there are no global functions.

Let's solve (2), by replacing functions by a basis $f_0, \ldots f_n \in H^0(X, L)$ where L is a line bundle. Using a local trivialization $L|_{U_i} \cong \mathcal{O}_{U_i}$ gives

$$\begin{tikzcd} V \cap U_i \arrow[r, "F_i"] \arrow[dr, "\varphi_i"'] & \mathbb{A}_k^{n+1} - (0) \arrow[d] \\ & \mathbb{P}_k^n \end{tikzcd}$$

It turns out that the φ_i will patch to define a morphism

$$X \supseteq V \xrightarrow{\varphi_L} \mathbb{P}_k^n$$

even though the F_i usually won't.

(1) is still an issue, but

Prop If L is generated by global sections, then φ_L is defined on all of X (In classical terminology, L is base point free.)

Def L is called very ample if it is generated by generated by global sections, and φ_L gives a closed immersion $X \hookrightarrow \mathbb{P}^n_k$.

Returning to curves, here is a concrete criterion

Thm If D is a divisor on genus g curve X, then

a) $\mathcal{O}_X(D)$ is generated by global sections if $\deg D \geq 2g$

b) $\mathcal{O}_X(D)$ is very ample if $\deg D \geq 2g+1$

pf We have an exact seq.

$$0 \to \mathcal{O}_X(D-p) \to \mathcal{O}_X(D) \to k(p) \to 0$$

which gives

$$H^0(\mathcal{O}_X(D)) \to H^0(k(p)) \to H^1(\mathcal{O}_X(D-p))$$

By Serre duality

$$H^1(\mathcal{O}_X(D-p)) \cong H^1(\Omega_X^1(-D+p))$$

$$\cong H^1(\mathcal{O}(K-D+p))$$

$$= 0$$

Since $\deg K - D + p < 0$

Therefore

$$H^0(\mathcal{O}_X(D)) \to H^0(k(p))$$

is surjective. This implies that there exist a section $\sigma \in H^0(\mathcal{O}_X(D))$ which doesn't vanish at p. Since this holds for all p, $\mathcal{O}_X(D)$ is globally generated.

The proof for (b) is similar. One has

$$0 \to \mathcal{O}_X(D-p-q) \to \mathcal{O}_X(D) \to k(p) \oplus k(q) \to 0$$
$$\text{or } \mathcal{O}_p/m_p^2$$

and checks last map on global section is surjective. This implies map $X \to \mathbb{P}^n$ injective and injection on tangent spaces //

Given a curve $X \subset \mathbb{P}^n_k$. Let $H \subset \mathbb{P}^n_k$ be a hyperplane such that $X \not\subset H$.
Then $X \cap H$ is a finite set.
Given $p \in X \cap H$, suppose the i-th coordinate of p is nonzero, then $p \in U_{x_i} = \{x_i \neq 0\}$
$U_i \cap H$ is defined by a linear polynomial $l \in k[\frac{x_0}{x_i}, \ldots \frac{x_n}{x_i}]$, we define the intersection multiplicity of H with X at p as $m_p = \operatorname{ord}_p l$ (this is independent of i)

Def

$$X \cdot H = \sum_{p \in X \cap H} m_p \, p$$

Def/lemma If H' is a 2nd hyperplane not containing X, $X \cdot H'$ is linearly equivalent to $X \cdot H$. The common degree is

$$\deg X = \deg X \cdot H = \deg X \cdot H'$$

Ex Let X be a genus 1 curve
Such a curve is called an elliptic curve. Let $p \in X$, $D = 3p$.

One has $\deg K = 2 \cdot 2 = 0$
So R.R again

$$\ell(D) - \underbrace{\ell(K-D)}_{\overset{\shortparallel}{0}} = \deg D + 1 - 1$$
$$= 3$$

Since $\deg < 0$

Therefore we have an embedding $X \subset \mathbb{P}^2_k$

Since $\deg D = 3$, X is an embedded as a cubic.