Chapter 5

Derived Functors and Tor

Refs.

1. Cartan, Eilenberg, Homological algebra
2. Grothendieck, Sur quelques points d'algèbre homologique
3. Mitchell, Theory of categories
4. Rotman, Intro to homological algebra.
5. Weibel, An introduction to homological algebra

5.1 Abelian categories

We start with some category theory. A category A is called abelian if it behaves like the category Mod_R. Rotman section 5.5 treats abelian categories in some detail. Most other books on homological algebra do as well. Let's write down a long list list of conditions on category A, which hold when $A = Mod_R$.

A1. $Hom_A(M, N)$ is an abelian group for every pair of objects M, N.

A2. Composition satisfies $f \circ (g + h) = f \circ g + f \circ h$ whenever both sides are defined. Similary, $(g + h) \circ f = g \circ f + h \circ f$ when this makes sense.

A3. There is a zero object satisfying $Hom_A(0, M) = Hom_A(M, 0) = 0$ for all M.

A4. For any pair of objects M, N we can form a direct sum, characterized up to isomorphism by $Hom(M \oplus N, T) = Hom(M, T) \oplus Hom(N, T)$ and $Hom(T, M \oplus N) = Hom(T, M) \oplus Hom(T, N)$

A5. Given a morphism $f : M \to N$, we can form an object $\ker f$ with a morphism $\ker f \to M$ characterized by $Hom(T, \ker f) = \ker Hom(T, M) \to Hom(T, N)$.

A6. Given $f : M \to N$, we can form an object $\operatorname{coker} f$ with a morphism $M \to \operatorname{coker} f$, characterized by $Hom(\operatorname{coker} f, T) = \ker Hom(N,T) \to Hom(M,T)$.

A7. Given $f : M \to N$, there exists an object $\operatorname{im} f$ with morphisms $M \to \operatorname{im} f$ and $\operatorname{im} f \to N$ such that their composition is f. We also require that $\operatorname{im} f$ is both $\operatorname{coker}(\ker f \to M)$ and $\ker(N \to \operatorname{coker} f)$. (A bit more precisely, these are canonically isomorphic.)

A category is called additive if A1-A4 hold, and it is called abelian if they all hold. The last axiom is the hardest to fathom. It is trying to capture the idea that in Mod_R, f can be factored through a surjective homomorphism $M \to \operatorname{im} f$ followed by an injective homomorphism $\operatorname{im} f \to N$. Since injectivity and surjectivity are not categorical notions, we replace them by saying that they are kernels or cokernels. To appreciate further subtleties, see example 5.5.

Example 5.2. *Mod_R is an abelian category.*

Example 5.3. *The category of finitely generated modules over a left noetherian ring is abelian. In particular, this applies to finitely generated abelian groups.*

Example 5.4. *The category of free abelian groups is additive but not abelian, because cokernels need not exist.*

Example 5.5. *The category of Hausdorff topological abelian groups and continuous homomorphisms satisfies A1-A6. The operations are the usual ones except for the cokernel. The cokernel of $f : M \to N$ in this category is the quotient $N/\overline{f(M)}$. However, if $f(M)$ is not closed, the map from $\operatorname{coker}(\ker f \to M) = M/\ker f$ to $\ker(N \to \operatorname{coker} f) = \overline{f(M)}$ is not an isomorphism. So A7 fails.*

Here is a simple yet powerful observation.

Proposition 5.6. *If A is abelian (resp. additive), then so is the opposite category A^{op}. This has the same objects as A but arrows are reversed, so that $Hom_{A^{op}}(N,M) = Hom_A(M,N)$.*

Proof. The axioms are self dual. □

Therefore

Example 5.7. *Mod_R^{op} is an abelian category. (NB: This should not be confused with $Mod_{R^{op}}$.)*

Given an abelian category, we can do most of what we have done so far in class. In particular, we can talk about exact sequences, injectives, projectives, complexes, and homology. We also note the following remarkable fact:

Theorem 5.8 (Freyd-Mitchell)**.** *Any small*[1] *abelian category can be embedded into a category modules over a ring in such a way that Hom's are the same, and exact sequences are the same.*

A proof can be found in Mitchell's book. Since we will mostly be working with explicit examples, we won't really need it. But it is reassuring to know that one can pretend that an abstract abelian category is a category of modules, without loosing too much. Also this means that various standard results such as the 5-lemma, snake lemma, etc. can be extended to an arbitrary abelian category.

In order to do more homological algebra, we need the following.

Definition 5.9. *An abelian category has enough injectives if for every object M, there exists an injective object I and a morphism $f : M \to I$ such that* $\ker f = 0$.

Example 5.10. *Mod_R has enough injectives.*

Example 5.11. *Mod_R^{op} has enough injectives. This is because an injective in Mod_R^{op} is a projective module, and every module is the quotient of a projective module.*

Example 5.12. *The category of finitely generated abelian groups does not have enough injectives.*

5.13 Derived functors

Definition 5.14. *A functor $F : A \to B$ between additive categories is called additive if $F(f + g) = F(f) + F(g)$, for every $f, g \in Hom(M, N)$.*

Lemma 5.15. *If F is additive, then $F(M \oplus N) \cong F(M) \oplus F(N)$.*

Proof. There are morphisms $i : M \to M \oplus N$, $j : N \to M \oplus N$, $p : M \oplus N \to M$, and $q : M \oplus N \to N$ such that $pi = id_M$, $qj = id_N$, $pj = 0$, $qi = 0$, and $ip + jq = id_{M\oplus N}$. The existence of such a collection of morphisms satisfying these relations characterizes the direct sum. The collection $F(i), \ldots$ would satisfy the same relations, therefore $F(M \oplus N)$ must be isomorphic to $F(M) \oplus F(N)$. □

Definition 5.16. *An additive (covariant) functor $F : A \to B$ from one abelian category from one category to another is left (right) exact if whenever*

$$0 \to M \to N \to P \to 0$$

[1] This is a set theoretic condition. In Gödel-Bernays, or similar set theory, one distinguishes between sets and classes. Classes are allowed to be very big, but sets are not. For example, one can form the class of all sets, but it wouldn't be a set. One is not allowed to form the class of all classes, thus avoiding the standard paradox of Cantor's set theory. A category is called small if the collection of the objects and morphisms form a set as opposed to a proper class.

is exact,

$$0 \to F(M) \to F(N) \to F(P)$$

(resp.

$$F(M) \to F(N) \to F(P) \to 0$$

is exact.) A functor which both right and left exact is called exact.

We can also handle contravariant functors $F : A \to B$ by treating them as covariant functors from $F : A^{op} \to B$. These are left or right exact if the second form is.

Let us fix a left exact functor $F : A \to B$, and let assume that A has enough injectives. An injective resolution of an object M is an exact sequence

$$0 \to M \to I^0 \to I^1 \dots$$

with I^i injective. By an argument dual to what we did for projective resolutions, we can see

Lemma 5.17. *Every M possesses an injective resolution.*

By arguments similar to what we did for Ext, we have

Theorem/Def 5.18. *We define the right derived functors*

$$R^i FM = H^i(F(I^\bullet))$$

The isomorphism class of these objects do not depend on the resolution.

Theorem 5.19. *R^iF extend to additive functors from $A \to B$ with $R^0F = F$. Given a short exact sequence*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

there is a long exact sequence

$$\dots R^iFM_1 \to R^iFM_2 \to R^iFM_3 \to R^{i+1}FM_1 \dots$$

Derived functors were introduced by Cartan and Eilenberg in their book in the mid 1950's in order to unify several disparate theories. Grothendieck carried the story further in his landmark paper shortly thereafter.

Example 5.20. *Fix a module N, and consider the left exact functor $Hom_R(-, N) : Mod_R^{op} \to Ab$. The right derived functors*

$$R^i Hom_R(-, N) = Ext^i_R(-, N)$$

by definition.

However, if we fix M, and consider $Hom_R(M, -) : Mod_R \to Ab$ we can also take derived functors. A much less obvious fact is

Theorem 5.21.

$$R^i Hom_R(M, -) \cong Ext^i_R(M, -)$$

Since Rotman does not appear to do this, we indicate the proof. A δ-functor is a sequence of functors $F^i : A \to B$ such that for any exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

there is a long exact sequence

$$\dots F^i M_1 \to F^i M_2 \to F^i M_3 \to F^{i+1} M_1 \dots$$

such that the connecting maps are natural in the appropriate sense. For example, the sequence of right derived functors $F^i = R^i F$ forms a delta functor. A functor F is called effacable if for any M, there exists an exact sequence $0 \to M \to I$ such that $F(I) = 0$.

Theorem 5.22. *Suppose that if F^i is a δ-functor such that for any $i > 0$ F^i is effacable. Then $F^i = R^i F^0$.*

Proof. This follows from the results of chap II sections 2.2-2.3 of Grothendieck. □

Proof of theorem 5.21. By results proved earlier $Ext^i(M, -)$ is a δ-functor. Furthermore if $i > 0$ and I is injective, $Ext^i(M, I) = 0$. Therefore $Ext^i(M, -)$ is effacable. So the result follows from the previous theorem. □

A right exact functor $F : A \to B$ is the same thing as a left exact functor $F' : A^{op} \to B^{op}$. So that we can take right derived of F'. When the story is translated back to F, we arrive at the notion of a left derived functor. To be explicit, given M, choose a projective resolution

$$\dots P_1 \to P_0 \to M \to 0$$

We need to assume that A has enough projectives to guarantee this exists. Set

$$L_i F = H_i(F(P_\bullet))$$

The key properties are:

- These are independent of the choice of resolution.
- These are additive functors from $A \to B$ such that $L_0 F = F$.
- A short exact sequence

  $$0 \to M_1 \to M_2 \to M_3 \to 0$$

  gives rise to a long exact sequence

  $$\dots L_i M_1 \to L_i M_2 \to L_i M_3 \to L_{i-1} M_1 \dots$$

5.23 Tor functors

Given a right R-module M, consider the functor $T : Mod_R \to Ab$ defined by

$$T(N) = M \otimes_R N$$

This is a right exact functor. We define

$$Tor_i^R(M, N) = L_i T$$

Then given

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

we get an exact sequence

$$\ldots Tor_1^R(M, N_1) \to M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$$

This will allow us to compute this in principle, but we still need a few more tricks.

Proposition 5.24. *If N is flat, then $Tor_i(M, N) = 0$ for $i > 0$.*

Proof. This follows from the construction

$$Tor_i(M, N) = L_i T(N) = H_i(P_\bullet \otimes N)$$

where $P_\bullet \to M \to 0$ is a projective resolution. Since N is flat,

$$\ldots P_1 \otimes N \to P_0 \otimes N \to M \otimes N \to 0$$

is exact. This means that $P_\bullet \otimes N$ has no homology in positive degrees.

□

Theorem 5.25. *Suppose that R is a (commutative) integral domain with field of fractions K. If $f \in R$ is nonzero,*

$$Tor_i(M, R/fR) = \begin{cases} \{m \in M \mid fm = 0\} & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

$$Tor_i(M, K/R) = \begin{cases} \{m \in M \mid \exists f \in R, fm = 0\} & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

Remark 5.26. *"Tor" is short for "torsion". The theorem partly explains why this name makes sense.*

Proof. We have an exact sequence

$$0 \to R \xrightarrow{f} R \to R/fR \to 0$$

Since R is flat, we obtain

$$0 = Tor_i(M,R) \to Tor_i(M,R/fR) \to Tor_{i-1}(M,R) = 0$$

for $i > 1$. We can identify $M \otimes R = M$ and the map $1 \otimes f$ with f. Therefore we also have

$$0 \to Tor_1(M,R/fR) \to M \xrightarrow{f} M$$

The proof of the second isomorphism is similar. We use the sequence

$$0 \to R \to K \to K/R \to 0$$

and the fact that K is flat. □

We note the following useful symmetry property.

Theorem 5.27. *Under the identification of left (right) R-modules with right (left) R^{op}-modules,*

$$Tor_i^R(M,N) \cong Tor_i^{R^{op}}(N,M)$$

In particular, if R is commutative,

$$Tor_i^R(M,N) \cong Tor_i^R(N,M)$$

Comments about the proof. The result is stated in theorem 7.1 in Rotman, but the proof given there is incomplete. What's missing is the fact that one can compute Tor using a projective resolution of the second variable. See theorem 2.7.2 of Weibel for this. (We may do this later, if there is time.) □

5.28 Homology of a group

Fix a group G and G-module i.e. $\mathbb{Z}G$-module M. We regard $\mathbb{Z}$ as a left (and also right) G-module with trivial G-action. Earlier we defined

$$H^i(G,M) = Ext^i_{\mathbb{Z}G}(\mathbb{Z},M)$$

In the current language, we could also define it as the right derived functors of the left exact functor

$$M \mapsto M^G$$

Recall that $M^G \subset M$ is the submodule of element invariant under G. It is the largest submodule on which G acts trivially. Let M_G be the largest quotient module on which G acts trivially. More explicitly

$$M_G = M/\{gm - m \mid m \in M, g \in G\}$$

Lemma 5.29. *Treating $\mathbb{Z}$ as a right $\mathbb{Z}G$-module with trivial G action,*

$$M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$$

Proof. We define a surjective ring homomorphism $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ by

$$\epsilon(\sum_i n_i g_i) = \sum_i n_i$$

Let $I = \ker \epsilon$. This is the two sided ideal generated by $g - 1$ with $g \in G$. Consider the exact sequence

$$0 \to I \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

Tensoring with M gives a sequence

$$I \otimes M \to \mathbb{Z}G \otimes M \to \mathbb{Z} \otimes M \to 0$$

We can identify the middle module with M, and the image of the first map with $\{(g-1)m \mid g \in G, m \in M\}$. So the lemma is now proved. □

Corollary 5.30. *$M \mapsto M_G$ is right exact.*

We define group homology by

$$H_i(G, M) = Tor_i^{\mathbb{Z}G}(\mathbb{Z}, M)$$

The lemma shows that

$$H_0(G, M) = M_G$$

Before describing the next result, we recall that the commutator (or derived) subgroup $[G, G] \subseteq G$ is the normal subgroup generated by all commutators $ghg^{-1}h^{-1}$. The quotient $G/[G, G]$ can be characterized as the largest abelian quotient of G.

Theorem 5.31. $H_1(G, \mathbb{Z}) \cong G/[G, G]$

Proof. With the above notation, we have an exact sequence

$$Tor_1^{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) \to Tor_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \to I \otimes_{\mathbb{Z}G} \mathbb{Z} \to \mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \xrightarrow{r} \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z} \to 0$$

By theorem 5.27

$$Tor_1^{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) \cong Tor_1^{\mathbb{Z}G^{op}}(\mathbb{Z}, \mathbb{Z}G)$$

In fact $g \mapsto g^{-1}$ induces an isomorphism between $\mathbb{Z}G$ and $\mathbb{Z}G^{op}$. Therefore

$$Tor_1^{\mathbb{Z}G^{op}}(\mathbb{Z}, \mathbb{Z}G) \cong Tor_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) = 0$$

because $\mathbb{Z}G$ is flat. The map marked r above can be identified with the identity $\mathbb{Z} \to \mathbb{Z}$. By definition $H_1(G, \mathbb{Z}) = Tor_1(\mathbb{Z}, \mathbb{Z})$. Therefore, we can conclude

$$H_1(G, \mathbb{Z}) \cong I \otimes_{\mathbb{Z}G} \mathbb{Z} = I \otimes_{\mathbb{Z}G} \mathbb{Z}G/I = I/I^2$$

Let $f : G \to I/I^2$ be given by $f(g) = g - 1 \, mod \, I^2$. Since

$$(gh - 1) - (g - 1) - (h - 1) = (g - 1)(h - 1) \in I^2$$

f is a homomorphism. Since I/I^2 is abelian, it factors through a homomorphism $\bar{f} : G/[G, G] \to I/I^2$. An explicit inverse is constructed on page 540 of Rotman, So $\bar{f}$ is an isomorphism. □