Hodge Modules

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Filtered holonomic $\mathcal{D}$-modules

Let X be a complex algebraic variety of dimension n.

Definition

A filtered regular holonomic $\mathcal{D}$-module with $\mathbb{Q}$ is a triple $M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$, consisting of the following objects:

1. A constructible complex of $\mathbb{Q}$-vector spaces K.
2. A regular holonomic right $\mathcal{D}_X$-module $\mathcal{M}$ with an isomorphism
$$DR(M) \simeq \mathbb{C} \otimes_{\mathbb{Q}} K.$$
By the Riemann Hilbert correspondance, this makes K a perverse sheaf.
3. A good filtration $F_\bullet M$ by $\mathcal{O}_X$-coherent subsheaves of $\mathcal{M}$ such that
    1. $F_p\mathcal{M} \cdot F_k\mathcal{D}_X \subseteq F_{p+k}\mathcal{M}$
    2. $gr^F_\bullet \mathcal{M}$ is coherent over $gr^F_\bullet \mathcal{D}_X$

Remark on the de Rham Complex

For a right $\mathcal{D}_X$-module $\mathcal{M}$, we have a left $\mathcal{D}$-module $\mathcal{N}$ such that $\omega_X \otimes \mathcal{N} = \mathcal{M}$.

With the following isomorphism,

$$DR(\mathcal{M}) \simeq DR(\mathcal{N})[n] =$$

$$[\mathcal{N} \to \Omega_X^1 \otimes \mathcal{N} \to \cdots \to \Omega_X^{n-1} \otimes \mathcal{N} \to \Omega_X^n \otimes \mathcal{N}][n]$$

When we have a filtered $\mathcal{D}$-module $\mathcal{M}$, we have

$$F_p\mathcal{M} = F_{p+n}\mathcal{N} \otimes_{\mathcal{O}_X} \omega_X.$$

We also have the natural filtered family of subcomplexes

$$F_pDR(\mathcal{M}) \simeq F_pDR(\mathcal{N})[n] =$$

$$[\mathcal{N}_p \to \Omega_X^1 \otimes \mathcal{N}_{p+1} \to \cdots \to \Omega_X^n \otimes \mathcal{N}_{p+n}][n]$$

Example

Let $\mathcal{M} = \omega_X$ with the following filtration

$$F_p\omega_X = \begin{cases} \omega_X & \textit{if } \ p \geq -n \\ 0 & \textit{if } \ p < -n \end{cases}$$

Then $(\omega_X, F_\bullet\omega_X, \mathbb{Q}[n])$ is a filtered regular holonomic $\mathcal{D}$-module with $\mathbb{Q}$-structure.

For $0 \leq p \leq n$ we have

$$F_{-p}DR(\omega_X) \simeq [0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \to \Omega_X^n][n]$$
$$gr^F_{-p}DR(\omega_X) \simeq \Omega_X^p[n-p]$$

Nearby and Vanishing Cycles

For a holomorphic function $f : X \to \Delta$ which is submersive over the punctured unit disk $\Delta^* = \Delta \backslash \{0\}$, we have following commutative diagram:

$$
\begin{array}{ccccc}
\tilde{X} & \xrightarrow{\pi} & X & \xleftarrow[i]{} & X_0 \\
\downarrow & & \downarrow f & & \downarrow \\
\mathbb{H} & \xrightarrow{e} & \Delta & \longleftarrow & \{0\}
\end{array}
$$

$\mathbb{H} =$ Upper half-plane
$\tilde{X} =$ the fiber product of X and $\mathbb{H}$ over Δ
$X_0 = f^{-1}(0)$

$$
\begin{array}{ccccc}
\tilde{X} & \xrightarrow{\pi} & X & \xleftarrow{\;i\;} & X_0 \\
\downarrow & & \downarrow f & & \downarrow \\
\mathbb{H} & \xrightarrow{e} & \Delta & \longleftarrow & 0
\end{array}
$$

Definition

Let K be a contsructible complex of $\mathbb{C}$-vector spaces on X. We have the following two complexes:

Complex of nearby cycles

$$\psi_f K = i^{-1} R\pi_*(\pi^{-1}K)$$

Vanishing cycles

$$\phi_f K = Cone(i^{-1}K \to \psi_f K)$$

Example

Suppose $f : X \to \Delta$ is proper and smooth on $X \backslash X_0$. If $x \in X_0$, then we have

$$\mathcal{H}^i(\psi_f K)_x \simeq \mathbb{H}^i(B^\circ_{\epsilon,x} \cap X_t; K|_{X_t})$$

$$\mathcal{H}^i(\phi_f K)_x \simeq \mathbb{H}^{i+1}(B^\circ_{\epsilon,x}, B^\circ_{\epsilon,x} \cap X_t; K|_{X_t})$$

$$\mathbb{H}^i(X_0; \psi_f K) \simeq \mathbb{H}^i(X_t; K|_{X_t})$$

Where $X_t = f^{-1}(t)$ for $0 < |t|$ sufficiently small and $B^\circ_{\epsilon,x}$ is an open ball of radius ϵ in X, centered at x.

Reference: L. Maxim, Intersection Homology & Perverse Sheaves

Recall:

- (Gabber) When K is perverse, the shifted complexes
$$ {}^p\psi_f K = \psi_f K[-1] \text{ and } {}^p\phi K = \phi_f K[-1] $$
are perverse sheaves.

- ${}^p\psi_f K$ has a monodromy operator T, induced by the automorphism $z \to z+1$ of the upper half-plane $\mathbb{H}$.
- Since perverse sheaves form an abelian category, we have the following decomposition
$$ {}^p\psi_f K = \bigoplus_{\lambda \in \mathbb{C}^\times} {}^p\psi_{f,\lambda} K $$
Where ${}^p\psi_{f,\lambda} K = ker(T - \lambda id)^m$, for $m \gg 0$, are the eigenspaces.

We have a similar decomposition for ${}^p\phi_f$.

V-filtrations

Let $f \in \mathcal{O}_X$ an arbitrary nontrivial function. For a filtered $\mathcal{D}_X$-module $M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$, we use the graph embedding

$$(id, f) : X \hookrightarrow X \times \mathbb{C}$$

to obtain a filtered $\mathcal{D}_{X\times\mathbb{C}}$-module $(\mathcal{M}_f, F_\bullet\mathcal{M}_f)$.

Where

$$\mathcal{M}_f = (id, f)_+\mathcal{M} = \mathcal{M}[\partial_t]$$

$$F_\bullet\mathcal{M}_f = F_\bullet(id, f)_+\mathcal{M} = \bigoplus_{i=0}^{\infty} F_{\bullet-i}\mathcal{M} \otimes \partial_t^i.$$

Definition

A <u>V- filtration</u> on $\mathcal{M}_f$ is a rational filtration $(V_\gamma = V_\gamma \mathcal{M}_f)_{\gamma \in \mathbb{Q}}$ that is exhaustive and increasing such that the following conditions are satisfied:

- Each V_γ is a coherent module over $\mathcal{D}_X[t, \partial_t t]$
- For each $\gamma \in \mathbb{Q}$ and $i \in \mathbb{Z}$, we have an inclusion

  $$V_\gamma \cdot V_i \mathcal{D}_{X \times \mathbb{C}} \subseteq V_{\gamma+i}.$$

  Furthermore, $V_\gamma \cdot t = V_{\gamma-1}$ for $\gamma < 0$.
- For every $\gamma \in \mathbb{Q}$, if we set $V_{<\gamma} = \bigcup_{\gamma' < \gamma} V_{\gamma'}$, then $t\partial_t - \gamma$ acts nilpotently on $gr_\gamma^V = V_\gamma / V_{<\gamma}$.

Recall:

- (Kashiwara, Malgrange) When $\mathcal{M}$ is regular holonomic and ${}^p\psi_f K$ is quasi-unipotent, the V-filtration for $\mathcal{M}_f$ exists and it is unique.
- (Kashiwara, Malgrange) The graded quotients $gr_k^V \mathcal{M}_f$ are again regular holonomic $\mathcal{D}$-modules on X whose support is contained in the original divisor $X_0 = f^{-1}(0)$.
- We endow each $\mathcal{D}_X$-module $gr_\gamma^V \mathcal{M}_f$ with the filtration induced by $F_\bullet \mathcal{M}_f$

$$F_p gr_\gamma^V \mathcal{M}_f = \frac{F_p \mathcal{M}_f \cap V_\gamma M_f}{F_p \mathcal{M}_f \cap V_{<\gamma} \mathcal{M}_f}$$

Definition

The unipotent nearby cycles D_X-module of $\mathcal{M}_f$ along t is defined as

$$\psi_{t,1}\mathcal{M} := gr^V_{-1}\mathcal{M}_f.$$

The vanishing cycles D_X-module of $\mathcal{M}_f$ along t is

$$\phi_{t,1}\mathcal{M} := gr^V_0\mathcal{M}_f$$

From the previous discussion and the definitions above, it seems as though we have the following to be filtered regular holonomic $\mathcal{D}$-modules with $\mathbb{Q}$-structure:

$$\psi_{f,1}M = (gr^V_{-1}\mathcal{M}_f, F_{\bullet-1}gr^V_{-1}\mathcal{M}_f, {}^p\psi_{f,1}K)$$

$$\phi_{f,1}M = (gr^V_0\mathcal{M}_f, F_\bullet gr^V_0\mathcal{M}_f, {}^p\phi_{f,1}K)$$

Definition

We say that $(\mathcal{M}, F_\bullet\mathcal{M}, K)$ is quasi-unipotent along $f = 0$ if all eigenvalues of the monodromy operator on ${}^p\psi_f K$ are roots of unity, and if the V-filtration $V_\bullet\mathcal{M}_f$ satisfies the following two additional conditions:

1. $t : F_pV_\gamma\mathcal{M}_f \to F_pV_{\gamma-1}\mathcal{M}_f$ is surjective for $\gamma < 0$.
2. $\partial_t : F_pgr^V_\gamma\mathcal{M}_f \to F_{p+1}gr^V_{\gamma+1}\mathcal{M}_f$ is surjective for $\gamma > -1$.

We say the $(\mathcal{M}, F_\bullet\mathcal{M}, K)$ is regular along $f = 0$ if $F_\bullet gr^V_\gamma\mathcal{M}_f$ is a good filtration for every $-1 \leq \gamma \leq 0$.

Theorem (Saito)

If $\mathcal{M}$ is holonomic, and is regular and quasi-unipotent along f, then we have

$$DR(gr_\gamma^V \mathcal{M}_f) \simeq \begin{cases} \psi_\lambda DR(\mathcal{M})[-1] & \text{for } -1 \leq \gamma < 0 \\ \phi_\lambda DR(\mathcal{M})[-1] & \text{for } -1 < \gamma \leq 0 \end{cases}$$

where $\lambda = e^{2\pi i \gamma}$. Furthermore, we have the following identification between perverse sheaves and $\mathcal{D}$-modules

$$ {}^p\psi_{f,1}K \underset{Var}{\overset{can}{\rightleftarrows}} {}^p\phi_{f,1}K \qquad \psi_{f,1}\mathcal{M} \underset{t}{\overset{\partial_t}{\rightleftarrows}} \phi_{f,1}\mathcal{M}$$

$$\frac{1}{2\pi i} log T_u = N = Var \circ can \Leftrightarrow t\partial_t$$

If $M = (\mathcal{M}, F_\bullet \mathcal{M}, K)$ is a filtered regular holonomic $\mathcal{D}$-module with $\mathbb{Q}$-structure which is quasi-unipotent and regular along f, then

$$\psi_f M = \bigoplus_{-1 \leq \gamma < 0} (gr_\gamma^V \mathcal{M}_f, F_{\bullet -1} gr_\gamma^V \mathcal{M}_f, {}^p\psi_{f, e^{2\pi i \gamma}} K)$$

$$\psi_{f,1} M = (gr_{-1}^V \mathcal{M}_f, F_{\bullet -1} gr_{-1}^V \mathcal{M}_f, {}^p\psi_{f,1} K)$$

$$\phi_{f,1} M = (gr_0^V \mathcal{M}_f, F_\bullet gr_0^V \mathcal{M}_f, {}^p\phi_{f,1} K)$$

are filtered regular holonomic $\mathcal{D}$-modules with $\mathbb{Q}$-structure on X whose support is contained in $X_0 = f^{-1}(0)$.

Remark:

Let $j : X \times \mathbb{C} \backslash X \times \{0\} \hookrightarrow X \times \mathbb{C}$ be the natural inclusion map.

- Suppose M has strict support Z and M is quasi-unipotent and regular along f. If the restriction of f to Z is not constant, then

$$F_p \mathcal{M}_f = \sum_{t=0}^{\infty} (V_{<0} \mathcal{M}_f \cap j_* j^* F_{p-i} \mathcal{M}_f) \partial_t^i$$

  provided that $\partial_t : F_p gr_{-1}^V \mathcal{M}_f \to F_{p+1} gr_0^V M_f$ is surjective.

- The equality implies M is uniquely determined by its restriction to $Z \backslash (Z \cap X_0)$.

Given a filtered regular holonomic $\mathcal{D}$-module with $\mathbb{Q}$-structure $M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$. When is M a Hodge Module?

- First, for any Zariski-open subset $U \subset X$ and $f \in \Gamma(U, \mathcal{O}_U)$, the restriction of M to U is quasi-unipotent and regular along $f = 0$.
- Second, Saito requires M to admit a decomposition by strict support, in the category of regular holonomic $\mathcal{D}$-modules with $\mathbb{Q}$-structure.

Theorem

Let M be a filtered regular holonomic $\mathcal{D}$ module with $\mathbb{Q}$ structure, and suppose that $(\mathcal{M}, F_\bullet\mathcal{M})$ is quasi-unipotent and regular along $f = 0$ for every locally defined holomorphic function f. Then M admits a decomposition

$$M \simeq \bigoplus_{Z \subseteq X} M_Z$$

by strict support, in which each M_Z is again filtered regular holonomic $\mathcal{D}$-module with $\mathbb{Q}$-structure, if and only if one has

$$\phi_{f,1}M = ker(Var : \phi_{f,1}M \to \psi_{f,1}M(-1)) \oplus im(can : \psi_{f,1}M \to \phi_{f,1}M)$$

for every locally defined holomorphic function f.

The problem of defining Hodge modules is reduced to defining Hodge modules with strict support on irreducible closed subvarieties Z.

Let Z be an irreducible closed subvariety of X. Saito uses a recursive procedure to define the following category.

$$HM_Z(X,w) = \left\{ \text{Hodge Modules on } X \text{ with strict support on } Z \text{ with weight } w \right\}$$

1. If Z is a point $x \in X$, then we have an equivalence of categories between Hodge Structures and Hodge Modules with strict support on x.

$$(i_x)_* : HS(w) \simeq HM_x(X,w)$$

2. If $d_Z > 0$, then M belongs in $HM_Z(X,w)$ if the following conditions hold:
   Let $f \in \Gamma(U, \mathcal{O}_U)$ and suppose $Z \cap U \not\subseteq f^{-1}(0)$, then we have
   $$gr^W_{i-w+1}\psi_f M_U,\ gr^W_{i-w}\phi_{f,1} M_U \in HM_{<d_Z}(U,i)$$
   Where W is the monodromy filtration of the nilpotent operator N on the nearby cycles of $\psi_f M$.
   $HM_{<d_Z}(U,i)$ is the direct sum of $HM_{Z'}(U,i)$ with Z' running over closed irreducible subvarieties of U with $d_{Z'} < d_Z$.

Definition

The category of Hodge modules of weight w on X has objects

$$HM(X,w) = \bigcup_{d \geq 0} HM_{\leq d}(X,w) = \bigoplus_{Z \subseteq X} HM_Z(X,w);$$

its morphisms are the morphisms of regular holonomic $\mathcal{D}$-modules with $\mathbb{Q}$-structure.

Polarizations

A polarization on a Hodge module $M \in HM(X, w)$ is a perfect pairing

$$S : K \otimes_{\mathbb{Q}} K \to \mathbb{Q}_X(n - w)[2n]$$

with the following properties:

1. It's compatible with the filtration. That is, it extends to an isomorphism $M(w) \simeq \mathbf{D}M$ in the category of Hodge modules.
2. For every summand M_Z in the decomposition of M by strict support, and for every locally defined holomorphic function $f : U \to \mathbb{C}$ this is not identically zero on $U \cap Z$, we have
$$^p\psi_f S \circ (id \otimes N^i)$$
is a polarization of $^P gr^W_{i-w+1} \psi_f M_U := ker(N^{i+1})$ (primitive part).
3. If $\dim M_Z = 0$, then S is induced by a polarization of Hodge structures in the usual sense.

Definition

We say a Hodge module is polarizable if it admits at least one polarization, and we denote by

$$HM^p(X,w) \subseteq HM(X,w) \text{ and } HM^p_Z(X,w) \subseteq HM_Z(X,w)$$

the full subcategories of polarizable Hodge modules.

Theorem (Properties)

1. *There are no nonzero morphism from an object in $HM^p(X, w_1)$ to an object $HM^p(X, w_2)$ if $w_2 > w_1$*
2. *The category $HM^p(X,w)$ is abelian and any morphism is strict.*
3. *The category $HM^p(X,w)$ is semi-simple.*

Structure Theorem

Theorem (Saito)

For any closed irreducible subvariety $Z \subseteq X$, the restriction to sufficiently small open subvarieties of Z induces an equivalence of categories

$$HM_Z^p(X, w) \simeq VHS_{gen}^p(Z, w - dimZ)$$

where the right-hand side is the category of polarizable variations of pure Hodge structures of weight $w - dimZ$ defined on a smooth dense open subvarieties U of Z. Moreover, we have a one-to-one correspondence between polarizations of $M \in HM_Z(X, w)$ and those of the corresponding generic variation of Hodge structure.

Structure Theorem

Theorem (Saito)

For any closed irreducible subvariety $Z \subseteq X$, the restriction to sufficiently small open subvarieties of Z induces an equivalence of categories

$$HM_Z^p(X, w) \simeq VHS_{gen}^p(Z, w - dimZ)$$

where the right-hand side is the category of polarizable variations of pure Hodge structures of weight $w - dimZ$ defined on a smooth dense open subvarieties U of Z. Moreover, we have a one-to-one correspondence between polarizations of $M \in HM_Z(X, w)$ and those of the corresponding generic variation of Hodge structure.

- $(\omega_X, F_\bullet \omega_X, \mathbb{Q}[n])$ is a polarizable Hodge module of weight n.

Direct Image Theorem

Theorem (Saito)

Let $f : X \to Y$ be a projective morphism of smooth complex algebraic varieties, and $M = (\mathcal{M}, F_\bullet, K) \in HM^p_Z(X, w)$. Let ℓ be the first Chern class of an f-ample line bundle. Then the direct image $f_*(M, F_\bullet)$ as a filtered $\mathcal{D}$-module is strict, and we have

$$\mathcal{H}^i f_* M := (\mathcal{H}^i f_*^{\mathcal{D}}(\mathcal{M}, F), {}^p\mathcal{H}^i f_* K) \in HM^p(Y, w + i)$$

together with isomorphisms

$$\ell^i : \mathcal{H}^{-i} f_* M \simeq \mathcal{H}^i f_* M(i)$$

Moreover, a polarization of M induces a polarization on $\mathcal{H}^i f_* M$ in the Hodge-Lefschetz sense.

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