V - filtration

D-modules on a Disk

1. Setup

Let Δ be a disk centered at 0

$\Delta^* = \Delta - \{0\}$

$j : \Delta^* \hookrightarrow \Delta$, $i : 0 \hookrightarrow \Delta$ inclusions

z the coordinate

$\widetilde{\Delta}^* \xrightarrow{p} \Delta$ the universal cover, which we can be identified with $\operatorname{im} t > 0$

$z = \exp(2\pi i t)$

Our goal is to describe regular holonomic (analytic) D-modules M on Δ.

Since M is generically an integrable connection, there is no loss in assuming $M|_{\Delta^*} = V$ is an integrable connection. Let us now start with a vector bundle V on Δ^* with a (integrable) connection ∇. V is necessarily trivial, so we can idtfy ∇ with

$$\frac{d}{dz} + A(z)$$

We assume ∇ is <u>regular</u> which means solutions to $\nabla f = 0$ have moderate growth near 0 (are $O(\frac{1}{|z|^n})$ for some n)

Fuch's criterion implies we can (and will) assume $A(z) = \underbrace{A_{-1}}\frac{1}{z} + A_0 + \cdots$

called residue of ∇.

At the moment V is $\mathcal{D}_{\Delta^*}$-module. We can extend it to a $\mathcal{D}_\Delta$-module by taking the direct image $j_* V$. However it would not be quasi-coherent. So we let

$$j_*^m V \subset j_* V$$

consist of sections with moderate growth near 0. This is a sub $\mathcal{D}$-module which is coherent on $\mathcal{O}_\Delta(*0)$, and hence quasi-coherent on $\mathcal{O}_\Delta$

2. Simple Objects

The category of regular holonomic modules on Δ is Artinian, so can get a good (partial) understanding by describing the simple objects. There are 3 types

1) $\mathcal{D}_{\mathbb{A}} / z\, \mathcal{D}_{\mathbb{A}} = \int_i^0 \mathbb{C}_0$.

2) $\mathcal{O}_{\mathbb{A}}$

3) Take a rank one vector bundle V with a connection $\frac{d}{dz} + \frac{a}{z}$, $a \notin \mathbb{Z}$.

Form $M = j_*^m V$

It is not hard to check these are simple. The fact that these are all of the simple objects is harder but follows from results stated previously about simple objects arising from intermediate extensions. Given a regular connection V on $\mathbb{A}^*_1$, the intermediate extension

$$j_{!*} V$$

is a regular holonomic $\mathcal{D}_{\mathbb{A}}$-module with no subquotients supported on 0. (For this reason, it's also called the minimal extension.)

3. Intermediate Extensions

We want to describe intermediate extensions in general. Naively, one might try $j_*^m V$, but it can have subquotients supported at 0.

To start from the beginning a solution to $\nabla f = 0$ is usually multivalued, i.e. it lives on $\tilde{\Delta}^*$. If $f_1, \dots, f_n$ is a basis of solutions, then

$$\vec{f}(t+1) = T\vec{f}(t)$$

for some matrix called monodromy.

Prop $T = \exp(-2\pi i \operatorname{Res}_0 \nabla)$

In writing the previous formulas, we implicitly chose an extension of V to a trivial vector bundle $\bar{V}$ and ∇ to an operator $\bar{\nabla} : \bar{V} \longrightarrow \Omega^1_\Delta(\log 0) \otimes \bar{V}$. The pair $(\bar{V}, \bar{\nabla})$ is not uniquely determined. The last proposition explains the ambiguity, which amounts to choosing $\log T$.

An extension $\bar{V}$ amounts to a choice of a suitable $\mathcal{O}_\Delta$-submodule of $j_*^m V$.

An explicit choice of $\bar{V}$ due to Deligne is the submodule $V^{\geq 0} \subset j_*^m V$ generated by sections with logarithmic growth, or more precisely which grows like $O((\log |z|)^N)$. This can also be characterized as the extension for which the logarithmic connection $\nabla^{\geq 0}$ has a residue with real parts of eigenvalues in $[0,1)$. More generally for each $r \in \mathbb{R}$, we can consider the extension $V^{\geq r}$ (resp $V^{>r}$) where real parts lie in $[r, r+1)$ (resp. $(r, r+1]$).

<u>Lemma</u> If $r \geq s$, then $V^{\geq r} \subseteq V^{\geq s}$ and $V^{>r} \subseteq V^{>s}$

Here is heuristic explanation:
The condition to be in $V^{\geq s}$ is that the fiber $V_0 / z V_0$ at is spanned by generalized eigenvectors for $z\partial$ with eigenvalues having real part in $(s, s+1)$.
If $v \in \Gamma(V^{\geq s})$ is the lift of any such eigenvector then, at least, formally $z^{r-s} v$ is an eigenvector with real part in $[r, r+1)$.

Thus we get an $\mathbb{R}$-indexed filtration on $j_*^m V$ called the Kashiwara-Malgrange or V-filtration. We'll say more later

<u>Thm</u> $j_{!*} V \subseteq j_*^m V$ is the D-module generated by $V^{>-1}$

4. Perverse sheaves on disk

To understand the structure of reg. holonomic modules on Δ, we can look their image under Riemann-Hilbert. Since Ω^1_Δ is trivial, we can identify the de Rham complex with

$$DR(M) = \underset{\substack{\uparrow \\ \deg -1}}{M} \xrightarrow{\partial} M = L$$

We see that

$$\text{st} \begin{cases} \mathcal{H}^i(L) = 0 & \text{for } i > 0 \\ \mathcal{H}^0(L) & \text{is supported at } 0 \end{cases}$$

Using the dual module $\mathbb{D}M$, we find that the same conditions hold for the Verdier dual.

$$\mathbb{D}L = R\mathcal{H}om(L, \mathbb{C}[-2])$$

A complex $L \in D^b_c(\Delta, \mathbb{C})$ with these properties is called a "perverse sheaf".

For example, if $\mathcal{L}$ is a local system on Δ^*, then $j_! \mathcal{L}[1]$, $j_* \mathcal{L}[1]$ and $Rj_* \mathcal{L}[1]$ are all perverse.

The structure of the category of Perverse sheaves is not easy to understand from the definition. A better description is by vanishing cycles. We will do this in part II