FROM CLASSICAL HODGE THEORY TO HODGE MODULES

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These are my notes for the spring school in Algebraic Geometry in Beijing, March 2015. My goal is to give a rapid overview of Hodge theory from the beginning to the notion of a Hodge module.

1. Classical Hodge theory

We start with a quick summary of classical Hodge theory. Proofs and further details can be found in, for example, Griffiths-Harris [GH] or Wells [W].

1.1. **Hodge and Lefschetz decompositions.** Let X be a smooth projective variety over $\mathbb{C}$ (with its classical topology). This implies that X is a compact oriented C^∞ manifold. So by de Rham's theorem, we can represent singular cohomology $H^i(X,\mathbb{C})$ by differential forms

Theorem 1.2 (de Rham)**.** *If $\mathcal{E}^i(X)$ denotes the space of complex valued C^∞-forms, then*

$$H^i(X,\mathbb{C}) \cong \frac{\{\alpha \in \mathcal{E}^i(X) \mid d\alpha = 0\}}{\{d\beta \mid \beta \in \mathcal{E}^{i-1}(X)\}}$$

The most natural proof of this from our point of view is by sheaf theory. We will assume that people are familiar with this, but it won't hurt to do a quick review. Given a sheaf $\mathcal{F}$ on a space X, we can associate a sequence of cohomology groups $H^i(X,\mathcal{F})$, $i = 0,1,2\ldots$ such that $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$ is the space of global section. The key point is that when X is paracompact and Hausdorff (and we will always assume this), sheaf cohomology can computed by

$$H^i(X,\mathcal{F}) = \frac{\ker[\Gamma(X,\mathcal{F}^i) \to \Gamma(X,\mathcal{F}^{i+1})]}{\mathrm{im}[\Gamma(X,\mathcal{F}^{i-1}) \to \Gamma(X,\mathcal{F}^i)]}$$

whenever $\mathcal{F} \to \mathcal{F}^\bullet$ is a *resolution by fine sheaves*. The power of this is given by the fact that we can choose different resolutions for different purposes. Returning to the case where X is an oriented manifold, Poincaré's lemma gives an exact sequence of sheaves

$$0 \to \mathbb{C}_X \to \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \ldots$$

where $\mathbb{C}_X$ is the sheaf of locally constant $\mathbb{C}$-valued functions on X, and $\mathcal{E}^p_X$ is the sheaf of p-forms. Since the latter sheaves are fine, this is a fine resolution. Therefore we get an isomorphism between de Rham cohomology and sheaf cohomology of $\mathbb{C}_X$. A different fine resolution gives an isomorphism between sheaf cohomology and singular cohomology. And this concludes the outline of proof.

Since X is in fact a complex manifold, we can decompose forms into (p,q)-type:

$$\mathcal{E}^i(X) = \bigoplus_{p+q=i} \mathcal{E}^{pq}(X) \tag{1}$$

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where $\alpha \in \mathcal{E}^{pq}(X)$ if in local analytic coordinates

$$\alpha = \sum f_{i_1\ldots i_p, j_1,\ldots,j_q} dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots d\bar{z}_{j_q}$$

Theorem 1.3 (Hodge decomposition)**.** *There is a bigrading*

$$H^i(X,\mathbb{C}) = \bigoplus_{p+q=i} H^{pq}(X)$$

where $H^{pq}(X)$ is the subspace represented by (p,q)-forms. Moreover $\overline{H^{pq}} = H^{qp}$.

We want to emphasize that this is not merely a formal consequence of (1). The theorem yields nontrivial topological restrictions on smooth projective varieties, which do not hold for general compact complex manifolds such as the Hopf manifold.

Corollary 1.4. *The odd Betti numbers $b_1, b_3, \ldots$ are even integers.*

Proof. $b_1 = 2\dim H^{10}$ etc. □

Here are the key ideas of the proof of the theorem.

(1) A Riemannian metric is a choice of C^∞ family of inner products g on the tangent spaces. If we choose a Riemannian metric on X, then we can define a Laplacian $\Delta = dd^* + d^*d$, where d^* is the adjoint with respect to the inner product determined by the metric. When X is compact and Riemannian, Hodge (and Weyl) proved that any cohomology class has a unique representative which is *harmonic* i.e. which lies in the kernel of Δ. In broad outline, this involves first proving the statement for L^2 forms using some functional analysis, and using the fact that (weak) L^2 solutions of Δ are C^∞ because it is elliptic.
(2) A metric g is *Kähler* if the complex structure J is orthogonal, which means $g(JX, JY) = g(X,Y)$, and the Kähler 2-form $\omega(X,Y) = g(X, JY)$ is closed. For such a metric, we have a so called Kähler identity $\Delta = 2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$ where $\bar\partial$ is the Cauchy-Riemann operator. This connects the theory of harmonic forms to complex geometry. In particular, it follows that a form is harmonic if and only if all of its (p,q) components are harmonic. This shows that the Hodge decomposition holds for a compact Kähler manifold.
(3) The only thing left is to show that a smooth projective variety X is Kähler. By definition X can be embedded into a complex projective space $\mathbb{P}^N_{\mathbb{C}}$ as a submanifold. $\mathbb{P}^N_{\mathbb{C}}$ carries a natural hermitian metric called the Fubini-Study metric. This metric, and its restriction to X are Kähler.

Fix an embedding $X \subset \mathbb{P}^N$. If $H \subset \mathbb{P}^N_{\mathbb{C}}$ is a hyperplane, the homology class $[X \cap H] \in H_2(X,\mathbb{Q})$ is well defined. By Poincaré duality, we can identify this with a cohomology class $[X \cap H] \in H^2(X,\mathbb{Q})$. As a de Rham class it is represented by the Kähler form of the Fubini-Study metric. Cup product with this class, or wedge product with the Käher form, will be denoted by L. Let $n = \dim X$ denote the dimension as a complex manifold (so $2n$ is the real dimension). The following is proved with the help of further Kähler identities.

Theorem 1.5 (Hard Lefschetz)**.**

(1) *We have isomorphisms*

$$L^i : H^{n-i}(X,\mathbb{Q}) \xrightarrow{\sim} H^{n+i}(X,\mathbb{Q})$$

(2) *Let* $P^{n-i}(X,\mathbb{Q}) = \ker L^{i+1} : H^{n-i}(X,\mathbb{Q}) \to H^{n+i+2}(X,\mathbb{Q})$. *This is called primitive cohomology. Then*
$$H^i(X,\mathbb{Q}) = P^i(X,\mathbb{Q}) \oplus LP^{i-1}(X,\mathbb{Q}) \oplus L^2P^{i-2}(X,\mathbb{Q}) \oplus \dots$$

This rather complicated statement has a number of important consequences. For example, it implies that the even (and odd) Betti numbers form an increasing sequence $b_0 \le b_2 \le b_4 \dots$ up to b_n. We will use it construct a certain quadratic form which will play an important role later. We define a pairing
$$Q(\alpha,\beta) = (-1)^{\frac{1}{2}k(k-1)} \int_X L^{n-k}\alpha \wedge \beta$$
on $H^k(X)$. This is symmetric if k is even, and skew symmetric if k is odd. Let C denote multiplication by i^{p-q} on H^{pq}. This is called the Weil operator.

Theorem 1.6 (Hodge-Riemann bilinear relations)**.**

(1) $Q(H^{pq}, H^{rs}) = 0$ *if* $(p,q) \neq (r,s)$.
(2) $Q(C-,-)$ *is symmetric positive definite on* $P^k(X,\mathbb{R})$

It is convenient to abstract things a bit. A *Hodge structure of weight* i consists of a finitely generated abelian group H together with decomposition
$$H_{\mathbb{C}} := H \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{pq}$$
such that $\overline{H^{pq}} = H^{qp}$. We sometimes refer to this as *pure* Hodge structure, to distinguish it from mixed Hodge structures introduced later. A rational Hodge structure is defined similarly by replacing the abelian group by a finite dimensional $\mathbb{Q}$-vector space. A choice of quadratic form Q satisfying the Hodge-Riemann relations (theorem 1.6 (1) and (2)) is called a polarization. So this theorem says that primitive cohomology carries a polarization. In fact, by combining this with the hard Lefschetz theorem, we can build a polarization on the full cohomology.

1.7. **Canonical Hodge structure.** Since the proof of the Hodge decomposition relied on a choice of Kähler metric, it is perhaps a bit surprising that it can be made independent of it. First we make a definition. A morphism of Hodge structures $f : H \to G$ of weight i is a homomorphism of abelian groups such that the extension to $\mathbb{C}$ takes H^{pq} to G^{pq}. In this way, we get a category of Hodge structures for each weight.

Theorem 1.8 (Deligne)**.** *There is a contravariant functor* $X \mapsto H^i_{del}(X)$ *from the category of smooth projective varieties over* $\mathbb{C}$ *to the category of polarizable Hodge structures of weight* i*, such that for any choice of Kähler metric on* X*,* $H^i_{del}(X)$ *is isomorphic to the Hodge structure given by harmonic forms.*

Before explaining the idea, we give an alternate definition which is often more convenient. Given a Hodge structure H of weight i, define the Hodge filtration by
$$F^p = H^{pq} \oplus H^{p+1,q} \oplus \dots$$
We have the following easy lemma.

Lemma 1.9. *For each* p*,* $H_{\mathbb{C}} = F^p \oplus \overline{F^{i-p+1}}$. *Conversely, any filtration satisfying the last condition arises from the unique weight* i *Hodge structure given by* $H^{pq} = F^p \cap \overline{F^q}$.

This becomes an equivalence of categories provided we declare morphisms in the second instance to be filtration preserving maps.

Given a smooth projective variety X, let $\mathcal{E}^\bullet(X)$ denote the de Rham complex. We filter this by

$$F^p\mathcal{E}^i(X) = \bigoplus_{i\geq p} \mathcal{E}^{r,i-r}(X)$$

One easily checks that this is a subcomplex, i.e. $dF^p\mathcal{E}^\bullet(X) \subset F^p\mathcal{E}^\bullet(X)$. Setting

$$F^pH^i(X) = \mathrm{im}[H^i(F^p\mathcal{E}^\bullet(X)) \to H^i(\mathcal{E}^\bullet(X))]$$

we have

Lemma 1.10. *$H^i(X)$ with the filtration $F^pH^i(X)$ is isomorphic to the space of harmonic forms with its Hodge filtration with respect any Kähler metric.*

Theorem 1.8 follows from this. It is convenient to say this in slightly fancier terms. The constant sheaf is resolved by the holomorphic de Rham complex[1]

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to \dots$$

Even though this is not a fine resolution, we can still draw the useful conclusion that $H^i(X,\mathbb{C})$ is isomorphic to the *hypercohomology* group $H^i(X,\Omega^\bullet_X)$. Recall that given a bounded complex of sheaves $\mathcal{F}^\bullet$, we can associate a group called the hypercohomology group $\mathbb{H}^i(X,\mathcal{F}^\bullet)$ with following properties:

(1) If $\mathcal{F}^\bullet$ consists of a single sheaf $\mathcal{F}^n$ concentrated in degree n, $\mathbb{H}^i(X,\mathcal{F}^\bullet) = H^{i-n}(X,\mathcal{F}^n)$.
(2) If $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a morphism of complexes, then we have an induced map $\mathbb{H}^i(X,\mathcal{F}^\bullet) \to \mathbb{H}^i(X,\mathcal{G}^\bullet)$
(3) If the map on complexes is a quasi-isomorphism, which means that it induces an isomorphism between $\mathcal{H}^i(\mathcal{F}^\bullet) \cong \mathcal{H}^i(\mathcal{G}^\bullet)$ for all i where

$$\mathcal{H}^i(\mathcal{F}^\bullet) = \frac{\ker[\mathcal{F}^i \to \mathcal{F}^{i+1}]}{\mathrm{im}[\mathcal{F}^{i-1} \to F^i]}$$

then the above map on hypercohomology is an isomorphism.
(4) If all of the sheaves $\mathcal{F}^i$ are fine, then hypercohomology is cohomology of the complex $\Gamma(\mathcal{F}^\bullet)$.

Returning to our original story, we can filter the holomorphic de Rham complex by the so called stupid filtration

$$F^p\Omega^j_X = \begin{cases} \Omega^j_X & \text{if } j \geq p \\ 0 & \text{otherwise} \end{cases}$$

This induces a filtration on hypercohomology

$$F^p\mathbb{H}^i(X,\Omega^\bullet_X) = \mathrm{im}[\mathbb{H}^i(X,F^p\Omega^\bullet_X) \to \mathbb{H}^i(X,\Omega^\bullet_X)]$$

Lemma 1.11. *Under the isomorphism $H^i(X,\Omega^\bullet_X) \cong H^i(X,\mathbb{C}) = H^i_{del}(X)_\mathbb{C}$, the filtration induced by $F^p\Omega^\bullet_X$ maps to the Hodge filtration of $H^i_{del}(X)$.*

[1]We use the symbols $\mathcal{O}_X, \Omega^1_X, \dots$ to stand for the sheaf of holomorphic functions, 1-forms etc.

2. Mixed Hodge structures

Deligne gave a far reaching generalization of classical Hodge theory to arbitrary, possibly singular or noncompact, complex algebraic varieties. He was partly motivated with the analogy of what happens over finite fields. We start with an example. If X is a projective curve with a single node. Let $\tilde{X}$ be the normalization. Topologically, X is obtained by identifying two points $p, q \in \tilde{X}$. One sees that X carries an additional loop corresponding to a path γ from p to q. It follows that the first Betti number $b_1(X) = b_1(\tilde{X}) + 1$ is odd, so $H^1(X)$ cannot carry a weight 1 Hodge structure. Nevertheless, we do have something resembling a Hodge structure. We define a basis of $H^1(X, \mathbb{Q}) = Hom(H_1(X, \mathbb{Z}), \mathbb{Q})$ to consist of γ^* together with a basis α_i^* of $H^1(\tilde{X}, \mathbb{Q})$ dual to a basis of $H_1(\tilde{X}, \mathbb{Z})$. Any holomorphic 1-form $\omega \in H^0(\tilde{X}, \Omega^1_{\tilde{X}})$ can be expanded as

$$\omega = \left(\int_\gamma \omega\right)\gamma^* + \sum \left(\int_{\alpha_i} \omega\right)\alpha_i^* \in H^1(X, \mathbb{C})$$

Let $F^1 \subset H^1(X, \mathbb{C})$ denote the span of the holomorphic 1-forms as above, and let $W_0 \subset H^1(X, \mathbb{Q})$ be the span of γ^*. Notice if we take the quotient $H^1(X)/W_0$, then we get $H^1(\tilde{X})$ and F^1 projects onto the correct subspace of it. Now we are ready for the general definition. A rational *mixed Hodge structure* consists of a finite dimensional $\mathbb{Q}$-vector space $H_\mathbb{Q}$, an increasing filtration $W_\bullet$ on $H_\mathbb{Q}$ called the weight filtration, and a decreasing filtration $F^\bullet$ on $H_\mathbb{C}$ called the Hodge filtration such that for each k, $Gr_k^W H = W_k/W_{k-1}$ with the filtration $\mathrm{im}(F^p \cap W_k \otimes \mathbb{C})$ is a pure Hodge structure of weight k. An integral mixed Hodge structure consists of all of the above, plus a finitely generated abelian group $H_\mathbb{Z}$ and an isomorphism $H_\mathbb{Z} \otimes \mathbb{Q} \cong H_\mathbb{Q}$.

Here are some examples.

Example 2.1. *Given a pure Hodge structure H of weight k, if we set $W_k = H_\mathbb{Q}$ and $W_{k-1} = 0$ then this becomes a mixed Hodge structure.*

Example 2.2. *If X is a nodal curve as above, then $H^1(X, \mathbb{Z})$ carries an integral mixed Hodge structure that we essentially described above. To finish the description, set $W_1 = H^1(X, \mathbb{Q})$, $F^0 = H^1(X, \mathbb{C})$ and $H_\mathbb{Z} = H^1(X, \mathbb{Z})$.*

As should be clear from this description, the mixed Hodge structure on $H^1(X)$ carries a lot of information which would be useful to understand. First of all, it determines the Hodge structure $H^1(\tilde{X})$ and therefore $\tilde{X}$ by Torelli's theorem:

Theorem 2.3 (Torelli)**.** *A smooth projective curve Y is determined by $H^1(Y)$, together with the quadratic form Q and the Hodge structure on it.*

This is usually stated as saying that Y is determined by the Jacobian variety $J(Y)$ as a polarized abelian variety. A proof of the last statement can be found in [GH]. Recall that as an abstract group $J(Y)$ is the divisor class group, which is the free abelian group $Div(Y)$ generated by Y modulo a suitable subgroup. After fixing a base point $y_0 \in Y$, the Abel-Jacobi map which sends $\sum n_i p_i \in Div(Y)$ to $\sum n_i \int_{y_0}^{p_i}$ in

$$\frac{H^1(Y, \mathbb{C})}{F^1 + H^1(Y, \mathbb{Z})}$$

induces an isomorphism of abstract groups. This endows $J(Y)$ with the structure of a complex torus, and eventually an abelian variety.

Returning to our nodal curve X, we see that $\gamma^* = \int_\gamma$ projects ont the class $[p-q] \in J(\tilde{X})$. With a bit more work, we obtain a Torelli theorem, due to Carlson, that X is determined by $H^1(X)$ as a mixed Hodge structure, together with a polarization on $H^1(X)/W_0$.

2.4. **Deligne's mixed Hodge structure.** We define the category of mixed Hodge structures, where the morphisms are $\mathbb{Q}$-linear maps preserving both filtrations. The basic structure is given by:

Theorem 2.5 (Deligne)**.** *The category of mixed Hodge structures is abelian, and the functors $(H, W, F) \mapsto W_k$ and $(H, W, F) \mapsto F^p$ are exact.*

To say that the category is abelian roughly means that it behaves very much like the category of modules over a ring. This is far from obvious, since the categories of filtered or bifiltered vector spaces are not abelian.

Theorem 2.6 (Deligne)**.** *There is a functor $X \mapsto H^i_{del}(X, \mathbb{Q})$ from the category of complex algebraic varieties to the category of mixed Hodge structures. The underlying vector space is just singular cohomology. When X is smooth and projective, this coincides with the pure Hodge structure given in theorem 1.8.*

We will simply write $H^i_{del}(X, \mathbb{Q}) = H^i(X, \mathbb{Q})$ below. We want to describe the filtrations in the important special case where X is quasiprojective and smooth. By definition X possesses a projective compactification $\bar{X}$. After applying Hironaka's theorem on resolution of singularities, we can assume $\bar{X}$ is smooth and the complement $D = \bar{X} - X$ is a divisor with simple normal crossings. This means that about any point of X, we can find local analytic coordinates $z_1, \ldots, z_n$, such that D is given by $z_1 z_2 \ldots z_k = 0$, and each irreducible component of D is nonsingular. Define $\Omega^1_{\bar{X}}(\log D)$ to be the sheaf of $\mathcal{O}_{\bar{X}}$-modules generated by the meromorphic differentials

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n$$

This is locally free of rank n. Let $\Omega^p_{\bar{X}}(\log D)$ denote the pth exterior power of $\Omega^1_{\bar{X}}(\log D)$. Given $\alpha \in \Gamma(U, \Omega^p_{X}(\log D))$, we can differentiate this as a meromorphic form, and it is easy to see that $d\alpha \in \Gamma(U, \Omega^{p+1}_{\bar{X}}(\log D))$. Therefore we have a complex of sheaves $\Omega^\bullet_{\bar{X}}(\log D)$. The first step is the appropriate de Rham theorem:

Theorem 2.7. *There is an isomorphism $H^i(X, \mathbb{C}) \cong \mathbb{H}^i(\bar{X}, \Omega^\bullet_{\bar{X}}(\log D))$.*

With this in hand we can describe the filtrations. The Hodge filtration $F^p H^i(X, \mathbb{C})$ is induced using the stupid filtration $F^p \Omega^\bullet_{\bar{X}}(\log D)$ as before. For the weight filtration, we first define

$$W_k \Omega^p_{\bar{X}}(\log D) = \Omega^k_{\bar{X}}(\log D) \wedge \Omega^{p-k}_{\bar{X}}$$

In other words, to be in W_k we allow products of at most k logarithmic differentials dz_i/z_i. Now we filter

$$W_{i+k} H^i(X, \mathbb{C}) = \mathbb{H}^i(X, W_k \Omega^\bullet_{\bar{X}}(\log D))$$

(note the shift on the left). A separate argument needs to be made to show that this is defined over $\mathbb{Q}$.

It is still not clear why this yields a mixed Hodge structure. To explain this, we make the further simplifying assumption that D is a smooth divisor. Then there is a residue map

$$\Omega^p_{\bar{X}}(\log D) \to \Omega^{p-1}_D$$

which sends $\frac{dz_1}{z_1} \wedge \alpha \mapsto \pm\alpha|_D$. This induces a short exact sequence

$$0 \to \Omega^\bullet_{\bar{X}} \to \Omega^\bullet_{\bar{X}}(\log D) \to \Omega^{\bullet-1}_D \to 0$$

which gives a long exact sequence of hypercohomologies

$$\ldots \mathbb{H}^i(\bar{X}, \Omega^\bullet_{\bar{X}}) \to \mathbb{H}^i(\bar{X}, \Omega^\bullet_{\bar{X}}(\log D)) \to \mathbb{H}^{i-1}(D, \Omega^\bullet_D) \to \mathbb{H}^{i+1}(\bar{X}, \Omega^\bullet_{\bar{X}}) \ldots$$

This coincides with the Thom-Gysin sequence

$$\ldots H^i(\bar{X}, \mathbb{C}) \to H^i(X, \mathbb{C}) \to H^{i-1}(D, \mathbb{C})(-1) \xrightarrow{\gamma} H^{i+1}(\bar{X}, \mathbb{C}) \to \ldots$$

The map labelled γ is the Gysin map which is Poincaré dual to the restriction. The notation (-1) is called a Tate twist; it indicates that the filtration has been shifted so that $H^{i-1}(D)(-1)$ has weight $i+1$ and also the lattice is multiplied by a factor of $\frac{1}{2\pi i}$. Now comes the key point. Our filtrations are set up so that

$$W_{i-1}H^i(X) = 0,\ W_{i+1}H^i(X) = H^i(X)$$
$$W_iH^i(X) \cong \operatorname{coker}[\gamma : H^{i-2}(D)(-1) \to H^i(\bar{X})]$$
$$H^i(X)/W_i \cong \ker[\gamma : H^{i-1}(D)(-1) \to H^{i+1}(\bar{X})]$$

where the isomorphisms are compatible with F. The right sides of the last two equations carry Hodge structures of the expected weight.

For a more detailed treatment see [PS].

2.8. **Limit mixed Hodge structure.** Suppose that we are given a family of varieties $f : X \to \Delta$ over a small disk with smooth fibres over $\Delta^* = \Delta - \{0\}$. Then the restriction of X to Δ^* is a fibre bundle, and we have a homotopy equivalence between X_0 and X. This leads to a map $sp : H^i(X_0) \cong H^i(X) \to H^i(X_t)$ for $t \neq 0$, called cospecialization which we would like to understand. As a topological space, the restriction of $X \to \Delta$ to the circle of radius $|t|$ is obtained by gluing the ends of $X_t \times [0,1]$ together. This gluing gives an automorphism $T : H^i(X_t, \mathbb{Q}) \to H^i(X_t, \mathbb{Q})$ called *monodromy*. It is not hard to show that the image of sp lies in the invariant part

$$H^i(X_t, \mathbb{Q})^T = \ker H^i(X_t, \mathbb{Q}) \xrightarrow{T-1} H^i(X_t, \mathbb{Q})$$

Much deeper is the surjectivity.

Theorem 2.9 (Local invariant cycle theorem)**.** *The map*

$$sp : H^i(X_0, \mathbb{Q}) \to H^i(X_t, \mathbb{Q})^T$$

is surjective.

This theorem, due to Clemens and Schmid, was deduced from Schmid's theorem [Sc] on the existence of the so called limit mixed Hodge structure. There is a special case which is fairly easy, and which goes back in some sense to the work of Picard and Lefschetz. This is the situation where a smooth projective curve X_t degenerates to a nodal curve X_0. We can choose a loop $\alpha_1(t)$ on X_t which shrinks to a point as $t \to 0$. We may complete this to a continuous family of cycles $\alpha_1(t), \beta_1(t), \ldots,$ forming a symplectic basis. In the limit the remaining cycles $\beta_1(0), \alpha_2(0), \ldots$ would give a basis of $H_1(X_0)$. Under specialization $H_1(X_t) \to H_1(X_0)$, $\alpha_1(t) \mapsto 0$, $\beta_1(t) \mapsto$

$\beta_1(0)$ etc. Thus the image of *sp* is spanned by $\beta_1(t)^*, \alpha_2(t)^*, \dots$ in the dual basis. The Picard-Lefschetz formula shows that this is also the space of invariant classes. This example also shows that the cospecialization map is *not* a morphism of mixed Hodge structures. However, it can be made into one provided we change the weight filtration on $H^1(X_t)$ to $W_0 = \langle \beta_1(t)^* \rangle$, $W_1 = \langle \beta_1(t)^*, \alpha_2(t)^* \dots \rangle$, $W_2 = H^1(X_t)$.

The last example hints at the general picture, but there is one important change. In general, we should not work with a specific X_t, but instead with an idealized "nearby fibre". This is the fibered product $\pi : \tilde{X} = \tilde{\Delta}^* \times_\Delta X \to X$ where $\tilde{\Delta}^* \to \Delta^*$ denotes the universal cover. It is the cohomology $H^i(\tilde{X})$, which is only noncanonically $H^i(X_t)$, that carries the limit mixed Hodge structure. Note that $H^i(\tilde{X})$ also carries a monodromy transformation which we also denote by T. Using Jordan canonical form, we can decompose $T = T_u T_s = T_s T_u$ where T_u is unipotent and T_s is semisimple. An important theorem of Borel, Grothendieck, Landman... shows that T_s has finite order, so that T is a so called quasi-unipotent transformation. The information contained in the unipotent part is equivalent to the nilpotent transformation $N = \log(T_u) = (T_u - I) - \frac{1}{2}(T_u - I)^2 + \dots$ Next comes a bit of linear algebra:

Theorem 2.10. *Given a finite dimensional vector space V over a field of characteristic 0 with a nilpotent endomorphism N and an integer m, there exists a unique increasing filtration $M_\bullet$ such that $NM_k \subset M_{k-2}$ and the "hard Lefschetz" isomorphism $N^k : Gr^M_{m+k} \cong Gr^M_{m-k}$ holds.*

M is called the monodromy (weight) filtration shifted by m. We apply this with $V = H^i(\tilde{X}, \mathbb{Q})$, N as above, and $m = i$.

Theorem 2.11 (Schmid). *There exists a mixed Hodge structure on $H^i(\tilde{X})$ with weight filtration M.*

The Hodge filtration is the limit $\lim_{t\to 0} F^p H^i(X_t)$, where the limit is taken in a sense that will be made precise in the next lecture. By Griffiths transversality, to be discussed later, $NF^p \subset F^{p-1}$. Therefore N preserves the Hodge and weight filtrations up to a shift of -1 and -2 respectively. If $H^i(\tilde{X})(1)$ denotes the mixed Hodge structure gotten by shifting the filtrations in this way, $N : H^i(\tilde{X}) \to H^i(\tilde{X})(1)$ becomes a morphism. Clemens and Schmid in fact proved that

$$H^i(X_0) \to H^i(\tilde{X}) \xrightarrow{N} H^i(\tilde{X})(1)$$

is an exact sequence of mixed Hodge structures.

2.12. **Hodge-Lefschetz structures.** Steenbrink [St] gave a geometric construction of the limit mixed Hodge structure, which gives further insight into it. Suppose that $Y = \bigcup Y_i = f^{-1}(0)$ is a reduced divisor with normal crossings. Then his construction shows that the associated graded of the limit mixed Hodge structure, can be understood in terms of the Hodge structures associated to the unions $Y^{(i)}$ of intersections of components of Y. We want to discuss one rather technical point, since we will need this later on. Steenbrink realized $H^*(\tilde{X}, \mathbb{C})$ as the cohomology of a filtered double complex constructed using logarithmic differential forms. The associated spectral sequence degenerates at the second page. In more explicit terms, he constructed a family of complexes (the first page of the spectral sequence, but written in a nonstandard way)

$$\dots H^i_j \to H^{j+1}_{i-1} \to \dots,$$

where

$$H_i^j = \bigoplus_k H^{j+n-i-2k}(Y^{(2k+i+1)})(-i-k),\ n = \dim X - 1$$

whose cohomology is the associated graded of $H^*(\tilde{X})$ for some filtration, say L. In fact, L turns out to be the monodromy filtration. To prove this, one should note that

(HL1) The spaces H_i^j carry Hodge structures of weight $n+i+j$.

and the differentials of the above complexes are compatible with this structure. Furthermore, Steenbrink constructed

(HL2) an operator $N : H_i^j \to H_{i-2}^j(-1)$ satisfying hard Lefschetz $N^i : H_{-i}^j \cong H_i^j(-i)$. The usual Lefschetz operator $\ell : H_i^j \to H_i^{j+2}(1)$ commutes with N and also satisfies hard Lefschetz $\ell^i : H_i^{-j} \cong H_i^j(j)$.

Moreover this is compatible with N on $H^i(\tilde{X})$, and $NL_i \subset L_{i-2}$. What remains is to show that the hard Lefschetz property persists for the cohomology of the complex H_*^*. Steenbrink's proof of this was incomplete, and fact the fix required new ideas. A polarized bigraded Hodge-Lefschetz structure of weight n is a collection $H_i^j, \ldots$ satisfying (HL1), (HL2) and the following

(HL3) A quadratic form Q such that such that $Q(-, N^i\ell^j -)$ gives a polarization on the "bi-primitive" part $\ker N^{i+1} \cap \ell^{j+1}$.

The following theorem can be used to complete the argument.

Theorem 2.13. *Suppose that $H = \bigoplus H_i^j$ is a polarized bigraded Hodge-Lefschetz module with a differential $d : H_i^j \to H_{i-1}^{j+1}$ commuting with ℓ and N and satisfying $\langle x, dy\rangle = \pm\langle dx, y\rangle$. Then the cohomology $\ker d/\operatorname{im} d$ carries an induced polarized Hodge-Lefschetz structure.*

Proof. See [S1, 4.2.2] or [GN, 4.5]. □

3. Variations of Hodge structures on a curve

3.1. **Variations of Hodge structure.** Given a family of smooth projective varieties $f : X \to Y$ over a complex manifold, we can consider the family of Hodge structures $\bigcup H^i(X_y)$, $X_y = f^{-1}(y)$. The first thing to note $X \to Y$ is a C^∞ fibre bundle by Ereshmann's theorem, this implies that the sheaf $R^i f_*\mathbb{Q}$ which is the sheaf associated to the presheaf $U \mapsto H^i(f^{-1}U, \mathbb{Q})$, is locally constant. It need not be constant however. The nontriviality is measured by the monodromy representation $\pi_1(Y, y) \to Aut(H^i(X_y, \mathbb{Q}))$. At this point, we should recall

Theorem 3.2 (Riemann-Hilbert I). *Let Y be a complex manifold and k a field. There is an equivalence between the categories of:*

1. *locally constant sheaves of k-vector spaces,*
2. *k-linear representations of the fundamental group,*
3. *and when $k = \mathbb{C}$, holomorphic vector bundles V with an integrable connections $\nabla : V \to \Omega^1_X \otimes V$.*

The last item needs a bit more explanation. A connection is a $\mathbb{C}$-linear map satisfying the Leibnitz rule $\nabla f v = df \otimes v + f\nabla v$. In local coordinates, ∇ is determined by the endomorphisms ∇_{∂_i} given by $\nabla v = \sum dx_i \otimes \nabla_{\partial_i} v$. Integrability is the condition that ∇_{∂_i} commute. Existence and uniqueness theorems for PDE guarantee that if ∇ is integrable, then $\ker\nabla$ is a locally constant sheaf of the same rank as V. Conversely, given a locally constant sheaf $\mathcal{L}$ of $\mathbb{C}$-vector spaces, $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{L}$ is a holomorphic vector bundle. It carries an integrable connection ∇ such that $\ker\nabla = \mathcal{L}$. Integrability has another interpretation which we recall. The sheaf of rings D_X of holomorphic differential operators is locally generated by $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ subject to the Weyl relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$ and $[x_i, \partial_j] = \delta_{ij}$. Integrability is precisely the condition for the action of vector fields on V, given by $\partial_i \cdot v = \nabla_{\partial_i} v$, to extend to a left D_X-module structure.

Returning to our example, $V = \mathcal{O}_X \otimes_{\mathbb{Q}} R^i f_*\mathbb{Q}$ carries an integrable connection ∇ such $\ker\nabla = R^i f_*\mathbb{C}$. This is called the Gauss-Manin connection. The Hodge filtrations form subbundles $F^p \subseteq V$. These are not generally stable under ∇. Instead we have a weaker property called Griffiths transversality:

$$\nabla(F^p) \subset \Omega^1_X \otimes F^{p-1}$$

This has a natural interpretation in the context of D-modules. The ring D_X has filtration by order. $F_k D_X$ is the subsheaf of operators with locally at most k partial derivatives. If we set $F_\ell V = F^{-\ell} V$. Then Griffiths transversality is just the compatibility condition

$$F_k D_X \cdot F_\ell V \subset F_{k+\ell} V \tag{2}$$

We now give a name to this sort of structure. A rational *variation of Hodge structure* of weight i on a manifold Y consists of a locally constant sheaf $\mathcal{L}$ of $\mathbb{Q}$-vector spaces, subbundle $F^p \subset V = \mathcal{O}_X \otimes \mathcal{L}$ satisfying Griffiths transversality such that for each y, $(\mathcal{L}_y, F^\bullet_y \subset V_y \cong \mathbb{C} \otimes \mathcal{L}_y)$ is a pure Hodge structure of weight i. For all the deeper properties, it is important to require the existence of a polarization, which a quadratic form Q on the local system $\mathcal{L}$ which gives a polarization, in the previous sense, on all the fibres.

3.3. **Variations of Hodge structure on the disk.** Let $\Delta \subset \mathbb{C}$ be a small disk around 0, and let $\Delta^* = \Delta - \{0\}$ with $j : \Delta^* \to \Delta$ the inclusion. Suppose that $(\mathcal{L}, F^\bullet, Q)$ is a polarized variation of Hodge structures of weight m on Δ^*. We have an integrable connection $\nabla : V \to \Omega_{\Delta^*} \otimes V$. This is known to have *regular singularities*. This means roughly that the solutions $\nabla v = 0$ grow at worst like $|t|^{-N}$ for some N where t is the local parameter. More precisely, we want the multivalued section v to satisfy $||v||^2 = Q(Cv, \bar{v}) \le |t|^{-N}$ on angular sectors $\{\theta_1 \le arg(t) \le \theta_2\}$. An equivalent statement, going back to Fuchs, is that there is a local basis v_i of V so that the connection matrix A, determined by $\nabla v_i = \sum a_{ij} v_j$, has simple poles. Or to put it another way, there exist a vector bundle $\bar{V} \subset j_* V$ on Δ with an operator $\bar{\nabla} : \bar{V} \to \Omega^1_\Delta(\log 0) \otimes \bar{V}$ such that $(\bar{V}, \bar{\nabla})$ extends (V, ∇). This does not uniquely determine the extension. If we expand $A = R\frac{dt}{t}$ + higher order terms, then it is known that $R = -\frac{1}{2\pi i}\log(T)$, where T is the monodromy transformation associated to the local system. But this leaves an ambiguity in the choice of the branch of the logarithm We can make a unique choice by further requiring the eigenvalues of the residue R to lie in a half open unit interval. Let V^b (respectively $V^{>b}$) denote the bundle corresponding to $[b, b+1)$ (respectively $(b, b+1]$) . Our preferred choice will be $\bar{V} = V^{>-1}$.

We now want to bring the Hodge filtration into play. Let $f^p = rank F^p$. Fix a reference vector space U with a quadratic form P such that $(\mathcal{L}_t, Q_t) \cong (U, P)$ for all $t \in \Delta^*$. The collection of Hodge structures F^pU on U polarized by P with $\dim F^pU = f^p$ is parameterized by a complex manifold D called a (Griffiths) period domain. This is a noncompact manifold on which the group $G_{\mathbb{R}} = Aut((U_{\mathbb{R}}, Q))$ acts transitively. We can embed D as an open subset of the smooth projective variety $\check{D}$ parametrizing flags on F^pU satisfying $\dim F^p = f^p$ and one of the Hodge-Riemann relations $Q(F^p, F^{m-p+1}) = 0$. We can identify the universal cover $\tilde{\Delta}^*$ with the upper half plane $\mathbb{H}$. The variation of Hodge structure $\mathcal{L}$ determines a holomorphic map from the universal cover $p : \mathbb{H} \to D$ called the period map. We can view the monodromy T as lying in $G_{\mathbb{R}}$. The transformation T is known to be *quasi-unipotent*. This means that after replacing Δ by a finite cover branched at 0, T becomes unipotent. Let us assume this for now, and let $N = \log T$. We define $\tilde{p}(t) = \exp(-tN)p(t)$. We have $p(t+1) = Tp(t)$, therefore $\tilde{p}(t)$ is invariant. Since $\check{D}$ is compact, $\tilde{p}$ extends to holomorphic map $\Delta \to \check{D}$. In particular, $\tilde{p}(0) \in \check{D}$ corresponds to a filtration on U. This is the Hodge filtration that is put on the limit mixed Hodge structure. This also implies that $F^p\bar{V} = j_*F^p \cap \bar{V}$ is a subbundle of $\bar{V}$. This is still true in general, assuming that T is only quasi-unipotent.

In preparation for the next lecture, let us view this through the lens of D-module theory. Since V has an integrable connection it is naturally a D_{Δ^*}-module. The sheaf j_*V becomes a left D_Δ module with the rule $\partial_t v_i = \sum \langle \partial_t, a_{ij} \rangle v_j$, where $\langle , \rangle$ denotes contraction of vector fields and 1-forms. However, it is very big. It is more useful to focus on the D_Δ-submodule $\tilde{V} \subset j_*V$ generated by $\bar{V} = V^{>-1}$. This D-module is sometimes called the *minimal extension* of V because it has no sub or quotient D-module supported on 0. This will be explained in the next lecture.

Consider the map $\partial_t : \tilde{V} \to \tilde{V}$. We can see it is surjective and that the kernel coincides with the sheaf $j_*\mathcal{L} \otimes \mathbb{C}$. Thus we have a quasi-isomorphism

$$j_*\mathcal{L} \otimes \mathbb{C} \cong (\tilde{V} \xrightarrow{\partial_t} \tilde{V})$$

We can view the complex on the right as a de Rham complex associated to $\tilde{V}$. It is better to write this in a more invariant way as

$$j_*\mathcal{L} \otimes \mathbb{C} \cong DR_u(\tilde{V}) := \tilde{V} \xrightarrow{\nabla} \Omega^1_\Delta \otimes_{\mathcal{O}} \tilde{V} \tag{3}$$

where the differential sends $\nabla(v) = dt \otimes \partial_t v$. (The subscript u, which is nonstandard, stands for "unshifted". The significance of this will be clear later.)

We extend F^p to a filtration of $\tilde{V}$ by the rule

$$F^p\tilde{V} = \sum_{i=0}^{\infty} \partial_t^i F^{p+i}\bar{V} \tag{4}$$

This formula guarantees that $\partial_t F^p\tilde{V} = F^{p-1}\tilde{V}$ as we would like. We note also that the sheaves $F^p\tilde{V}$ are coherent over $\mathcal{O}_\Delta$, although $\tilde{V}$ itself is not.

3.4. **Zucker's Hodge decomposition.** Suppose that $(\mathcal{L}, F^\bullet V, Q)$ is a polarized variation of Hodge structure of weight k on a smooth projective curve X. We have a fine resolution

$$0 \to \mathcal{L} \otimes \mathbb{C} \to \mathcal{E}^0 \otimes V \xrightarrow{\nabla} \mathcal{E}^1 \otimes V \xrightarrow{\nabla} \mathcal{E}^2 \otimes V \to 0$$

Therefore the cohomology of $\mathcal{L} \otimes \mathbb{C}$ can be represented by V-valued differential forms. We use the polarization combined with a Kähler metric on X to introduce an inner product on such forms. This allows us to define the Laplacian $\Delta = \nabla\nabla^* + \nabla^*\nabla$ which is again elliptic. The Hodge theorem applied to this shows that the cohomology of $\mathcal{L} \otimes \mathbb{C}$ is isomorphic to $\ker \Delta$.

As a C^∞ bundle, V can be decomposed as $V = \bigoplus_{p+q=k} V^{pq}$ where $V^{pq} = F^p/F^{p+1}$. This introduces a multigrading on forms which gets reduced to a bigrading as follows:

$$(\mathcal{E}^0 \otimes V)^{pq} = \mathcal{E}^{00} \otimes V^{pq}$$
$$(\mathcal{E}^1 \otimes V)^{pq} = \mathcal{E}^{10} \otimes V^{p-1,q} \oplus \mathcal{E}^{01} \otimes V^{p,q-1}$$
$$(\mathcal{E}^2 \otimes V)^{pq} = \mathcal{E}^{11} \otimes V^{p-1,q-1}$$

We formally split $\nabla = D' + D''$, where D' is of type $(1,0)$ and $(0,1)$ with respect to this bigrading. A refinement of the earlier Kähler identity shows that $\Delta = 2(D''(D'')^* + (D'')^*D'')$, so it preserves the bigrading. Combining these observations leads to:

Theorem 3.5 (Deligne)**.** *$H^i(X, \mathcal{L})$ carries a pure polarized Hodge structure of weight $i + k$.*

Unfortunately, one is rarely in a situation where this can be applied. In practice, one usually has an $\mathcal{L}$ defined on a Zariski open subset $j : U \to X$. In order to get a Hodge theorem, we need to work with V-valued forms with L^2 growth conditions. Since L^2 cohomology, unlike ordinary cohomology, is sensitive to the metric, we need to choose it with care. Zucker chooses it to be asymptotic to the Poincaré metric around each puncture. With this choice, he proves that the sheaf of locally L^2-forms is a fine resolution of $j_*\mathcal{L} \otimes \mathbb{C}$. So that:

Theorem 3.6 (Zucker)**.** *$H^i(X, j_*\mathcal{L})$ carries a pure polarized Hodge structure of weight $i + k$.*

Also using appropriate Kähler identities, one gets a hard Lefschetz theorem (cf [KK]).

Theorem 3.7. *Cup product with the Kähler class induces an isomorphism*

$$H^0(X, j_*\mathcal{L}) \to H^2(X, j_*\mathcal{L})$$

We need a more explicit description of the Hodge filtration. Here we use the D-module approach of Saito. Let $\tilde{V}$ be the minimal extension of V to a D_X-module. We filter this as in (4). We can also filter the de Rham complex by

$$F^p DR_u(\tilde{V}) = F^p\tilde{V} \to \Omega^1_X \otimes F^{p-1}\tilde{V} \tag{5}$$

Theorem 3.8 (Saito-Zucker)**.** *Under the isomorphism $H^i(X, j_*\mathcal{L})\otimes\mathbb{C} \cong H^i(DR_u(\tilde{V}))$, the Hodge filtration on the left coincides with the filtration induced by* (5).

These results – without D-modules – are proved in [Z]. A nice account of this from the present viewpoint can be found in Sabbah [Sb1].

4. Hodge modules on a curve

4.1. **D-modules and perverse sheaves on curves.** At this point, we need to use the language of derived categories. Very roughly, we want to work with complexes up to quasi-isomorphism. Let X be a topological space and A a commutative ring. An object of $D^b(X, A)$ is a bounded below complex of sheaves of A-modules with finitely many nonzero cohomology groups. A morphism from $A^\bullet \to B^\bullet$ is represented by a diagram

$$A^\bullet \overset{\sim}{\leftarrow} C^\bullet \to B^\bullet$$

where the first arrow is a quasi-isomorphism. To complete the description, we would have to say when two diagrams represent the same morphism, and also how to compose. These details can be found in [GeM] for example. Here a few examples to keep in mind. Any sheaf of A-modules gives an object in $D^b(X, A)$. Suppose we are given a continuous map $f : Y \to X$ of sufficiently nice topological spaces and a sheaf $\mathcal{F}$ of A-modules on X such that $R^i f_*\mathcal{F} = 0$ for $i \gg 0$ (this will hold in all cases we consider). Then we let $\mathbb{R}f_*\mathcal{F}^\bullet \in D^b(X, A)$ denote $f_*\mathcal{F}^\bullet$ for some fine resolution $\mathcal{F}^\bullet$ of $\mathcal{F}$. This is well defined up to isomorphism.

Now suppose that X is a smooth projective curve. We want to describe a special class of D_X-modules called regular holonomic modules. We start with the basic building blocks.

Example 4.2. *Given a point $i : p \to X$, local parameter t (although it is independent of this) and a finite dimensional $\mathbb{C}$-vector space H, we define*

$$i_+H_p = \bigoplus_{n=0}^{\infty} \partial_t^n H_p$$

where H_p is the skyscraper sheaf at p and ∂_t is just treated as a symbol. This becomes a D_X-module where t acts trivially on elements of H and ∂_t is applied formally.

The next class of examples comes from the following proposition.

Proposition 4.3. *Let (V, ∇) be a vector bundle with an integrable connection with regular singularities on a Zariski open set $j : U \to X$. There exists a unique D_X-submodule $\tilde{V} \subset j_*V$ which restricts to V and which has no sub or quotient modules supported on the complement $D = X - U$. This is called the minimal extension of V.*

A D_X-module is called *regular holonomic* if it has a finite filtration such that the successive quotients are isomorphic to modules in one of the above two classes (4.2, 4.3). Let us say that the module is *decomposable* if it is a direct sum of such modules.

Given a D-module M we can form the de Rham complex $DR_u(M)$ as we did in (3). People usually shift this so that $DR(M) := DR_u(M)[1]$ has M in degree -1, because it makes it more symmetric under duality. We define an object of $D^b(X, \mathbb{C})$ to be a (decomposable) *perverse sheaf* if it arises as $DR(M)$ with M a (decomposable) regular holonomic module. Note that for the sake of expedience, we are turning what is usually stated as theorem (the Riemann-Hilbert correspondence) into a definition. We refer to [HTT] for the detailed story. We define an object of $D^b(X, \mathbb{Q})$ to be perverse if it perverse after tensoring by $\mathbb{C}$. By definition, perverse sheaves are built from the following basic examples.

Example 4.4. *If $M = i_+H_p$, then we see almost immediately that $DR(M) = H_p$.*

Example 4.5. *Let $M = \tilde{V}$ be as in proposition 4.3, then $DR(M) = j_*\mathcal{L}[1]$, where $\mathcal{L} = \ker\nabla$.*

The structure of perverse sheaves is summarized by:

Theorem 4.6. *The full category $Perv(X) \subset D^b(X, \mathbb{Q})$ of perverse sheaves is abelian. Any perverse sheaf has a finite filtration such that the successive quotients are either $\mathbb{Q}_p$ for some p, or $j_*\mathcal{L}[1]$ where $\mathcal{L}$ is a locally constant sheaf of finite dimensional $\mathbb{Q}$-vector spaces on a Zariski open $j : U \to X$.*

Corollary 4.7. *A perverse sheaf has at most two nonzero cohomology sheaves $\mathcal{H}^0$ and $\mathcal{H}^{-1}$ and the first is supported on a finite set and the second is locally constant on the complement of a finite set.*

The corollary together with a dual condition gives a complete characterization of perverse sheaves. For decomposable perverse sheaves the characterization is even easier. They are exactly sums of objects in the above two classes (4.4, 4.5). Not all perverse sheaves are decomposable.

Example 4.8. *Given a Zariski open $j : U \to X$, $\mathbb{R}j_*\mathbb{Q}_U[1]$ is a perverse sheaf which is not decomposable.*

4.9. **Kashiwara-Malgrange filtration.** We want to say a little more about how the minimal extension of proposition 4.3 is constructed when (the local system associated to) V has quasi-unipotent monodromy. This is the important case for Hodge theory. We localize to the case of a disk $\Delta \subset X$ such that $\Delta^* \subset U$. Given a rational number b, let V^b be the vector bundle on X such that (V, ∇) extends to $\bar{\nabla} : V^b \to \Omega^1_X(\log D) \otimes V^b$ such that the real parts of the eigenvalues of residues lie in $[b, b+1)$. We define $V^{>b}$ in the same way except that we take the interval as $(b, b+1]$. Let us make this more explicit. Let T be the local monodromy about a puncture $p \in D$. Decompose it as a product $T = T_sT_u = T_uT_s$, where T_u is unipotent and T_s is semisimple. Let $\{v_j\}$ be a local basis of (multivalued) solutions to $\nabla v = 0$ which diagonalizes T_s, i.e. $T_sv_j = \lambda_jv_j$. Then, if t is a local coordinate at p, V^b is locally generated by sections

$$\exp\left(-\frac{\log t \log T_u}{2\pi i}\right) t^{\alpha_j} v_j \in \Gamma(\Delta, j_*V)$$

where $\alpha_j \in [b, b+1)$ and $\exp(-2\pi i\alpha_j) = \lambda_j$. Likewise for $V^{>b}$. We can see that as subsheaves of j_*V, we have inclusions $V^{>b'} \subset V^{b'} \subset V^b$ whenever $b' > b$. We define $\tilde{V} \subset j_*V$ to be the D_X-module generated by $V^{>-1}$.

Proposition 4.10. *$\tilde{V}$ has no no sub or quotient module supported at 0 i.e. annihilated by t.*

Proof. Since the connection matrix can be assumed to be upper triangular, we have a filtration on V such that the successive quotients have rank one. Using this, we can reduce to the case where V has rank one. So we can identify $V^{>-1} = \mathcal{O}_\Delta t^\alpha$, where $\alpha \in (-1, 0]$ and ∂_t acts in the usual way. Then $\tilde{V} = \mathcal{O}_\Delta$ if $\alpha = 0$, and $\tilde{V} = \mathcal{O}_\Delta[t^{-1}]t^\alpha$ otherwise. In either case $\tilde{V}$ is a simple D_Δ-module supported on Δ, and this implies the proposition. $\square$

The previous proposition also follows from more general considerations that we want to explain. The module $\tilde{V}$ inherits a decreasing filtration $V^b\tilde{V} = \tilde{V} \cap V^b$, indexed by $\mathbb{Q}$, called the *Kashiwara-Malgrange* filtration. Such filtrations exist more generally, and are characterized by the properties $tV^\alpha M \subset V^{\alpha+1}M$, $\partial_t V^\alpha M \subset V^{\alpha-1}M$, and $V^\alpha M/V^{>\alpha}M$ is a finte dimensional space on which $t\partial_t - \alpha$ acts nilpotently. Define the unipotent nearby cycle, nonunipotent nearby cycle and unipotent vanishing cycle functors on the class of modules possessing such a filtration by

$$\psi_t^u M = V^0 M/V^{>0}M$$
$$\psi_t^{\neq u} M = \bigoplus_{-1<\alpha<0} V^\alpha M/V^{>\alpha}M$$
$$\phi_t^u = V^{-1}M/V^{>-1}M$$

These functors really do depend on t, but when it is understood we suppress it from the notation. We can get some sense of what information these functors give about the module from the next couple of examples.

Example 4.11. *$V^\alpha\mathcal{O}_\Delta = t^{[\alpha]}\mathcal{O}_\Delta$ if $\alpha \geq 0$ and $V^\alpha\mathcal{O}_\Delta = \mathcal{O}_\Delta$ if $\alpha < 0$. So that $\psi^u\mathcal{O}_\Delta = \phi^u\mathcal{O}_\Delta = \mathbb{C}$ and $\psi^{\neq u}\mathcal{O}_\Delta = 0$.*

Example 4.12. *If M is the module of example 4.2, then*

$$V^\alpha M = \bigoplus_{0\leq n\leq -1-\alpha} \partial_t^n H_p \tag{6}$$

so that $V^\alpha M = 0$ if $\alpha > -1$, $\psi^u M = \psi^{\neq u}M = 0$ and $\phi^u M = H$.

Proposition 4.13. *A D_Δ-module M has no sub or quotient module supported at 0 if only the following two conditions hold:*

1. *The map $can : \psi^u M \to \phi^u M$ induced by $-\partial_t$ is surjective.*
2. *The map $var : \phi^u M \to \psi^u M$ induced by t is injective.*

Proof. We prove one direction assuming the exactness of the above functors. If $N \subseteq M$ is supported at 0. The formulas in example 4.12 show that $t\phi^u(N) \subset \psi^u(N) = 0$. Therefore $\phi^u N \subseteq \phi^u M$ lies in the kernel of var. If N is the quotient of M supported at 0, then $\phi^u N$ is a quotient of coker can for similar reasons. $\square$

We can use this to give another proof of proposition 4.10. It also implies the next proposition.

Proposition 4.14. *A regular holonomic D_Δ-module M is decomposable if and only if $\phi^u M = \operatorname{im} can \oplus \ker var$.*

The above functors have topological interpretations (which is where the names come from). For example, $\psi^u M$ and $\psi^{\neq u}M$ are respectively isomorphic to the kernel and cokernel of $T_s - I$ on the space of multivalued flat sections $\{v \in \Gamma(\tilde{\Delta}^*, M) \mid \partial_t v = 0\}$.

4.15. **Hodge modules on curves (simple definition).** A Hodge module is a generalization of a variation of Hodge structure so as to allow singularities. The underlying vector bundle with connection is replaced by a D-module. The precise set up is as follows. Let us say that a pre-Hodge module on a curve X consists of a tuple $\mathcal{M} = (M, F^\bullet, \mathcal{L}, \alpha)$ where M is regular holonomic D_X-module, $F^\bullet$ is a filtration of M by $\mathcal{O}_X$-submodules satisfying (2) such that F^p/F^{p+1} is coherent, and $\mathcal{L}$ is a perverse sheaf over $\mathbb{Q}$ with an isomorphism $\alpha : \mathcal{L} \otimes \mathbb{C} \cong DR(M)$. Here are some basic examples.

Example 4.16. *Given a pure polarizable Hodge structure H and a point $i : p \to X$, we can associate a pre-Hodge module supported at p. The D-module is*

$$i_+H = \bigoplus \partial_t^n H$$

with filtration

$$F^p(i_+H) = \bigoplus \partial_t^n F^{p+n} \tag{7}$$

The perverse sheaf is $i_*H_{\mathbb{Q}}$.

Example 4.17. *Given a pure polarizable variation of Hodge structure $(V, F^\bullet, \mathcal{L})$ on a Zariski open set $j : U \to X$, the minimal extension $\tilde{V}$ gives a D-module with filtration given by (4). Together with the perverse sheaf $j_*\mathcal{L}[1]$, we get a pre-Hodge module.*

We define a (polarizable) Hodge module to be a pre-Hodge module which can be written as direct sum of modules in the previous two classes (4.16, 4.17). So the underlying D-module is decomposable. The collection of pre-Hodge modules becomes category where morphisms are compatible families of maps between pairs of perverse sheaves and pairs of D-modules, where the D-module maps should preserve filtrations. The full subcategory of Hodge modules is easily seen to be abelian, although the bigger category is not.

We can define the weight of a Hodge module to be k if all the summands of type 4.16 arise from weight k Hodge structures and all the summands of type 4.17 arise from weight $k-1$ variations of Hodge structure. As a corollary of Zucker's theorem we obtain.

Corollary 4.18. *If $\mathcal{L}$ is a perverse sheaf which is part of a Hodge module of weight k, then $H^i(X, \mathcal{L})$ carries a Hodge structure of weight $i + k$.*

4.19. **Hodge modules on curves (actual definition).** We were really cheating in defining Hodge modules the way we did. This was not Saito's original definition, but rather a consequence. The point really is that Saito was working in arbitrary dimensions. The naive definition we gave does not generalize in any obvious way. We describe Saito's conditions for a fixed pre-Hodge module $(M, F^\bullet, \mathcal{L}, \alpha)$. First of all, we need conditions which guarantee that the V and F filtrations behave well together. For every local coordinate t, Saito's requires that

(HM1) $\mathcal{L}$ should be quasi-unipotent about $t = 0$, so we have a V-filtration on M.
(HM2) $t : F^pV^\alpha M \xrightarrow{\sim} F^pV^{\alpha+1}M$ for $\alpha > -1$.
(HM3) $\partial_t : F^pV^\alpha M/V^{>\alpha}M \xrightarrow{\sim} F^{p-1}V^{\alpha-1}M/V^{>\alpha-1}M$ for $\alpha < 0$.

We want to say a few words about these rather technical conditions (HM2) and (HM3). These hold trivially for M in example 4.16 using (6) and (7). These conditions also hold – less trivially – for example 4.17. In fact, for a module generated by $V^{>-1}$, these are equivalent to (4) (cf [S1, prop 3.2.2]).

Next, Saito requires a refinement of the conditions of proposition 4.14 to hold in the filtered setting.

(HM4) For every local parameter t, $Gr_V^{-1}(M, F) = \operatorname{im} can \oplus \ker var$ as filtered modules.

The last two axioms are inductive axioms, where he uses induction on support.

(HM5) A pre-Hodge module with zero dimensional support is a Hodge module if and only if it arises from Hodge structure as in example 4.16.

The final axiom is rather subtle. Roughly speaking, we want restriction of $\mathcal{M} = (M, F^\bullet, \ldots)$ to any point $p \in X$ to be a Hodge structure; however, this is too naive for a couple of reasons. First of all, instead of restriction we need to use nearby/vanishing cycle functors with respect to some parameter t at p. Secondly, when we apply ψ_t^u we are exactly in the situation of Schmid's theorem, and the resulting object is really a mixed Hodge structure. We need to apply the associated graded with respect to the monodromy filtration relative to $N = \frac{1}{2\pi i}\log T_u$ to get pure Hodge structures.

(HM6) The associated graded objects of $\psi_t^u\mathcal{M}$ with respect to the monodromy filtration associated to N is a pure Hodge structure of expected weight. Similar statements should apply to $\phi_t^u\mathcal{M}$ and $\psi_t^{\neq u}\mathcal{M}$.

Thus far we have described possibly nonpolarizable Hodge modules. In order to get a really useful theory, we need to require polarizations. A polarization is a pairing on the D-module which satisfies certain inductive conditions. Roughly speaking, we want the notion to agree with the usual one for Hodge structures, and be compatible with nearby and vanishing cycle functors in an appropriate sense. We omit the precise details. With a few modifications, this gives the definition of polarizable Hodge modules for smooth varieties of arbitrary dimension. Saito even allows the ambient variety to be singular, essentially by taking modules on a larger smooth variety supported on the original variety. To simplify terminology we assume that all Hodge modules from now on are polarizable. The structure of this category is given by the next result.

Theorem 4.20 (Saito)**.** *Let X be a projective variety, and $Z \subset X$ an irreducible closed subvariety. A polarizable variation of Hodge structure on a Zariski open subset of Z extends to a Hodge module supported on Z such that the underlying D-module is the minimal extension. Every simple Hodge module is of this form. Every Hodge module is a direct sum of simple modules.*

When X is a curve, we recover the description of Hodge modules given in the previous section.

4.21. **Direct image.** We want to consider one basic example. Let $f : Y \to X$ be a surjective map from a smooth projective variety onto a curve. Let $n = \dim Y$ and let $\mathcal{M} = (\mathcal{O}_Y, F^\bullet, \mathbb{Q}_Y[n], \alpha)$ be the pre-Hodge module on Y, where the Hodge filtration is $F^0 = \mathcal{O}_Y, F^1 = 0$ and α is the isomorphism with the de Rham complex

$$\mathbb{C}_Y \cong \underbrace{\mathcal{O}_Y \to \Omega_Y^1 \to \ldots}_{DR_u(\mathcal{O}_Y)}$$

translated by n. Since $\mathcal{O}_Y$ is a simple D_Y-module with trivial filtration, it is not difficult to check that this is a Hodge module.

When f is smooth, we have already seen that $R^i f_*\mathbb{Q}$ is part of a polarizable variation of Hodge structure, and therefore it is gives rise to a Hodge module. We want to redo the construction so that it works even when f has singularities. The first step to take the direct image of $\mathcal{O}_Y$ in the category of D-modules. Before writing down formulas, we should explain the big picture. We want a complex of regular holonomic modules $\mathbb{R}f_+\mathcal{O}_Y$ (or more accurately an object in the appropriate derived category) such that $DR(\mathbb{R}f_+\mathcal{O}_Y) = \mathbb{R}f_*\mathbb{Q}[n]$. Taking ith cohomology yields a regular holonomic module that we denote by $R^i f_+\mathcal{O}_Y$. We can apply DR again to $R^i f_+\mathcal{O}_Y$ to obtain a perverse sheaf which we denote by ${}^p\mathcal{H}^i\mathbb{R}f_*\mathbb{C}[n]$. This is the same as $R^{n-1+i}f_*\mathbb{C}[1]$ when f is smooth but not in general. This begs the question: is there are direct construction of this perverse sheaf that avoids going through the Riemann-Hilbert correspondence? The answer is yes, and is provided by the theory of perverse t-structures [BBD, HTT]. To make a long story short, we get a sequence of functors ${}^p\mathcal{H}^i, i \in \mathbb{Z}$, from subcategory of $D^b(X,\mathbb{Q})$, called the constructible derived category, to $Perv(X)$ with certain nice properies. One important consequence of all of this is that we obtain a natural rational perverse sheaf ${}^p\mathcal{H}^i\mathbb{R}f_*\mathbb{Q}[n] \in Perv(X)$ such that $({}^p\mathcal{H}^i\mathbb{R}f_*\mathbb{Q}[n]) \otimes \mathbb{C} = {}^p\mathcal{H}^i\mathbb{R}f_*\mathbb{C}[n]$.

We now give the explicit construction. This can be done in two steps. First we factor f as the inclusion $\iota : Y \to Y \times X$ given by the graph followed by the projection $p : Y \times X \to X$. The direct image under the inclusion

$$\iota_+\mathcal{O}_Y = \bigoplus_{j=0}^{\infty} \partial^j \iota_*\mathcal{O}_Y$$

Then

$$R^i f_+\mathcal{O}_Y = \mathbb{R}^i f_*(\Omega^\bullet_{Y\times X/X} \otimes \iota_+\mathcal{O}_Y[n])$$

This is obviously an $\mathcal{O}_X$-module, and less obviously a D_X-module (I assume this will covered in other lectures). This is filtered by

$$F^p\iota_+\mathcal{O}_Y = \bigoplus_{j=0}^{-p} \partial^j\mathcal{O}_Y$$

$$F^pR^i f_+\mathcal{O}_Y = \operatorname{im}\mathbb{R}^i f_*(\Omega^\bullet_{Y\times X/X} \otimes F^{p-\bullet}\iota_+\mathcal{O}_Y[n])$$

Together with ${}^p\mathcal{H}^i\mathbb{R}f_*\mathbb{Q}[n]$ this forms a pre-Hodge module, and in fact a Hodge module. These statements are far from trivial using either definition of Hodge module.

5. Conclusion

5.1. **Decomposition theorem.** We now come to one of the main applications which is a new proof and refinement of the decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber [BBD]. Their original proof used characteristic p methods.

Theorem 5.2 (Saito)**.** *Let $f : X \to Y$ be a morphism of complex projective varieties. If $\mathcal{L}$ is a perverse sheaf which is part of a Hodge module, then $\mathbb{R}f_*\mathcal{L}$ decomposes in $D^b(X, \mathbb{Q})$ as a direct sum of translates of perverse sheaves which arise from Hodge modules.*

In particular:

Corollary 5.3. *If X is smooth, then $\mathbb{R}f_*\mathbb{Q}$ decomposes in $D^b(Y, \mathbb{Q})$ as a direct sum of translates of perverse sheaves.*

A detailed discussion of the theorem and various applications can be found in de Cataldo and Migliorini [dCM]. We note that the same authors have found a different Hodge theoretic proof of the last result. As a corollary [BBD, cor 6.2.9], we get a strong form of the local invariant cycle theorem.

Corollary 5.4. *Let $U \subset Y$ be the complement of the discriminant, $y_0 \in Y$, $B \subset Y$ is a small ball centered at y_0, if $y_1 \in U \cap B$, then the restriction*

$$H^i(X_{y_0}, \mathbb{Q}) \to H^i(X_{y_1}, \mathbb{Q})^{\pi_1(U \cap B)}$$

is surjective.

It had been conjectured for some time that the intersection cohomology of Goresky-Macpherson [GM] should carry a natural Hodge structure. In terms of D-module theory, the complex that computes intersection cohomology is the regular holonomic module given by the minimal extension of $\mathcal{O}$ on the smooth locus. This is part of a Hodge module.

Corollary 5.5. *The cohomology of the perverse sheaf underlying a pure Hodge module carries a pure Hodge structure. In particular, intersection cohomology carries a pure Hodge structure.*

Earlier we indicated that we can construct certain perverse sheaves ${}^p\mathcal{H}^i\mathbb{R}f_*\mathcal{L}$ corresponding to direct image D-modules to curves. In fact, this works in any dimension. The key point for proving the decomposition theorem is to establish the hard Lefschetz theorem for perverse sheaves (also originally due to [BBD] in the geometric case).

Theorem 5.6 (Saito)**.** *Let $f : X \to Y$ be a morphism of projective varieties. If $\mathcal{L}$ is a perverse sheaf which is part of a Hodge module, then cup product with the ith power of an ample class induces an isomorphism*

$$\ell^i : {}^p\mathcal{H}^{-i}\mathbb{R}f_*\mathcal{L} \xrightarrow{\sim} {}^p\mathcal{H}^i\mathbb{R}f_*\mathcal{L}$$

Combined with Deligne theorem [D1], this would give

$$\mathbb{R}f_*\mathcal{L} = \bigoplus {}^p\mathcal{H}^i\mathbb{R}f_*\mathcal{L}[-i] \tag{8}$$

We know that the perverse sheaves on right side come from Hodge modules and this will imply theorem 5.2.

So the only thing left to do is to prove theorem 5.6. We give a broad outline of the proof when X is a surface, Y is a smooth curve and $\mathcal{L} = \mathbb{Q}[\dim X]$ based on what we asserted (but didn't prove!) in the previous lecture. This will at least give a sense of how the theory works. To begin with, let us assume that $n = \dim X - 1$ is arbitrary. We can assume that f is surjectve, since it is trivial otherwise. We can make things more explicit. If $j : U \to Y$ is the complement of the discriminant D of f, then

$$ {}^p\mathcal{H}^{-i}\mathbb{R}f_*\mathcal{L} = j_*j^*R^{n-i}f_*\mathbb{Q}[1] \oplus \bigoplus_{p\in D} K_p^{-i} \tag{9}$$

where the sheaf K_p^i is a skyscraper sheaf supported at p. Product with ℓ^i will respect these decompositions. The usual hard Lefschetz theorem applied to the fibres implies that

$$\ell^i : j_*j^*R^{n-i}f_*\mathbb{Q}[1] \xrightarrow{\sim} j_*j^*R^{n+i}f_*\mathbb{Q}[1]$$

So we just have to prove that

$$\ell^i : K_p^{-i} \to K_p^i \tag{10}$$

is an isomorphism. At this point, vanishing cycles come into play. In rough terms, K_p^* comes from the part of the cohomology of $f^{-1}(p)$ not appearing in the nearby fibres. (To avoid getting overwhelmed with notation, we will be somewhat imprecise about indices and weights for the remaining discussion.) To make this precise, let t be a local parameter at p, and g its pull back to X. By the axioms, we can decompose $\phi_g^u\mathcal{L} = \operatorname{im} can \oplus \ker var$. Let M correspond to the second factor. We have an isomorphism $H^*(M) \cong K_p^*$. Thus we need to establish the hard Lefschetz for the cohomology of M. The perverse sheaf M is equipped with a nilpotent endomorphism N. Again from the axioms, the associated graded of M, with respect to the monodromy filtration W associated to N, is a Hodge module. Now let us suppose that $n = 1$. Then $Gr^W M$ is supported on the curve $f^{-1}(p)$. After pulling back to the normalization, we can apply theorem 3.7 to conclude that the hard Lefschetz holds for $H^*(Gr^W M)$. This is not quite what we want. To finish we first need to observe that by standard homological algebra, there is a spectral sequence

$$E_1 = H^*(Gr^W M) \Rightarrow H^*(M)$$

The first page E_1 together with the polarized Hodge structure on it (theorem 3.6) and the actions of ℓ and N constitutes a complex of polarized bigraded Hodge-Lefschetz structures. Thus theorem 2.13 shows that $E_2 = H^*(E_1)$ is a also a Hodge-Lefschetz structure, and in particular it satisfies hard Lefschetz with respect to ℓ. The final step is the to show that this degenerates at E_2, again for Hodge theoretic reasons. So that $E_2 = Gr^W H^*(M)$ satisfies hard Lefschetz. Then the same goes for $H^*(M)$ by linear algebra.

5.7. **Further developments.** We do not have time to discuss subsequent developments, so we merely mention them. Saito also developed a theory of mixed Hodge modules which are to Hodge modules what mixed Hodge structures are to pure Hodge structures [S2]. Mochizuki, building on earlier work of Sabbah, Saito and Simpson, has taken these ideas further with his work on twistor D-modules [Sb2].

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