Chapter 1

Manifolds

1.1 Topological Manifolds

In imprecise terms, a manifold is a space which looks locally like Euclidean space. Actually there several kinds. Let's start with the most basic.

Definition 1.1.1. *A topological manifold of dimension n (or n-manifold) is a metrizable topological space such point has a nbhd U homeomorphic to an open ball of $\mathbb{R}^n$. We refer to U as a coordinate nbhd, and U with a fixed homeomorphism $\phi : U \to V \subset \mathbb{R}^N$ as a chart.*

Recall that metrizable means that it comes from a metric. The condition ensures that the underlying space is sufficiently nice, but it can be omitted at this point. Note that any connected component of a manifold is also a manifold, so we usually just study the connected ones. And if I forget to say "connected manifold" , you should assume that's what I meant.

Example 1.1.2. *Any open subset of $\mathbb{R}^n$ is clearly an n-manifold.*

Example 1.1.3. *The n-sphere $S^n = \{(x_1, \ldots, x_n) \mid x_1^2 + \ldots + x_n^2 = 1\}$ is manifold. Given $p = (1, 0, \ldots 0)$, the hemisphere $S^n \cap \{x_1 > 0\}$ is a coordinate nbhd, with ϕ given by stereographic projection.*

Example 1.1.4. *If X and Y are manifolds of dimension n and m, then $X \times Y$ is manifold of dimension $n+m$. In particular, the torus $T^n = S^1 \times \ldots S^1$ (n times) is an n-manifold.*

Example 1.1.5. *The cone defined by $z^2 = x^2 + y^2$ in $\mathbb{R}^3$ is* **not** *a manifold. Why not?*

The next example is really important in algebraic geometry. So we study it in some detail.

Example 1.1.6. *Complex projective space $\mathbb{CP}^n = \mathbb{P}^n_{\mathbb{C}}$ is the set of complex lines through the origin (= one dimensional complex subspaces) in $\mathbb{C}^{n+1}$. Alternatively, let $t \in \mathbb{C}^*$ act on $\mathbb{C}^{n+1}$ by multiplications. Then $\mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$.*

We give $\mathbb{P}^n_{\mathbb{C}}$ the quotient topology, i.e. U is open iff it's preimage $\mathbb{C}^{n+1} - \{0\}$ is open.

Lemma 1.1.7. *$\mathbb{P}^n_{\mathbb{C}}$ is a manifold of dimension $2n$ (NB: algebraic geometers generally use complex dimension, which would be n.)*

Proof. Given $p = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} - \{0\}$, let $[p]$ denote the corresponding point in projective space. The variables z_i are called homogenous coordinates. Let $U_i = \{[z_0, \ldots, z_n] \mid z_i \neq 0\}$. This forms an open cover. The map

$$[z_0, \ldots, z_n] \mapsto (z_1/z_0, \ldots z_n/z_0)$$

defines a homeomorphism $U_0 \cong \mathbb{C}^n$. A similar construction applies to all U_i. □

The manifold $\mathbb{P}^n_{\mathbb{C}}$ is compact (why?). By the above proof it is a compactification of $\mathbb{C}^n$. If one wants to compactify $\mathbb{C}^n$ to a complex manifold (to be defined later). Then this is a simplest way to do it.

Finally, we describe a few more examples constructed by "cut and paste".

Example 1.1.8. *If one glues the ends of $[0, 1] \times \mathbb{R}$ by identifying $(0, x)$ with $(1, x)$, one gets $S^1 \times \mathbb{R}$. However, identifying $(0, x)$ with $(1, -x)$ results in the Moebius strip.*

Example 1.1.9. *Take two copies of T^2, say T_1, T_2. Choose open coordinate disks $D_i \subset T_i$. Note that boundaries $\partial \bar{D}_i$ are circles. Glue $T_1 - D_1$ to $T_2 - D_2$ along the circles $\partial \bar{D}_i$. This construction is called a connected sum, and denoted by $T^2 \# T^2$. This forms a new 2-manifold called a genus 2 surface. It can be visualized by drawing a 2 holed donut. This can be repeated several times. The manifold $T^2 \# T^2 \# \ldots T^2$ (g times) is a called a genus g surface.*

1.2 C^∞-manifolds

It is possible to do calculus on manifolds, but first we have to refine the definition.

Definition 1.2.1. *A C^∞ manifold of dimension n is a topological manifold X equipped with a collection of charts (called an atlas) $\phi_i : U_i \to V_i \subset \mathbb{R}^n$ such that $\phi_i \circ \phi_j^{-1}$ are C^∞, and of course $X = \bigcup U_i$.*

If we write $\phi_i(p) = (x_1(p), \ldots, x_n(p))$. The functions $x_1, \ldots$ are called local coordinates. We are requiring that new coordinates can be expressed as C^∞ functions of old coordinates, and visa versa. We say that two atlases are equivalent if their union is an atlas. Then to be a bit more pedantic, a C^∞ manifold is given by an equivalence class of atlases. (Alternatively, some authors will tell you pick the maximal one by Zorn's lemma.)

Example 1.2.2. *Open subsets of $\mathbb{R}^n$, S^n, T^n and $\mathbb{P}^n_{\mathbb{C}}$ are all C^∞ manifolds.*

For the last item, note that homogenous coordinates are *not* coordinates in the sense of the previous paragraph, but the ratios $z_0/z_i, z_2/z_i \ldots$ on U_i are. On $U_i \cap U_j$, we have two systems of coordinates z_k/z_i and z_k/z_j related by multiplying by z_i/z_j or its inverse.

The Moebius strip and the last example above $T^2\#\ldots$ can also be made into C^∞ manifold if the gluing is done with care. For the rest of this section manifold means C^∞ manifold.

We can produce many examples, with the help of the implicit function theorem from calculus. In the simplest form it says that if $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^∞ such that $f(0) = 0$ and $\frac{\partial f}{\partial x_{n+1}}(0) \neq 0$, and if

$$f(x_1, \ldots, x_{n+1}) = 0$$

then we can "solve for" x_{n+1} in terms of the previous variables, at least near the origin. Here is a more precise and stronger statement.

Theorem 1.2.3. *Suppose that $0 \in U \subseteq \mathbb{R}^{n+m}$ is open, and $f : U \to \mathbb{R}^m$ is a C^∞ function such that $f(0) = 0$ and $(\frac{\partial f_i}{\partial x_{n+i}}(0))_{i=1,\ldots m}$ is invertible. Then $f^{-1}(0)$ is the graph of another C^∞ function g near the origin. More precisely, there exists open sets $0 \in V \subseteq \mathbb{R}^n, 0 \in W \subseteq \mathbb{R}^m$ and C^∞ $g : V \to W$ such that $V \times W \subset U$, and*

$$\forall (p, q) \in V \times W, f(p, q) = 0 \Leftrightarrow q = g(p)$$

Corollary 1.2.4. *Suppose that $f : U \to \mathbb{R}^m$ is C^∞ such that $X = f^{-1}(0) \neq \emptyset$ and the Jacobian $(\partial f_i/\partial x_{n+i})$ is invertible along all points of X Then X has the structure of C^∞ n-manifold.*

Sketch. Suppose $p_1 \in X$, which for simplicity we assume is 0. Then the implicit function theorem produces V, W, g as above. Set $U_1 = (V \times W) \cap X$. Then the projection $\phi_1 : U_1 \to V = V_1$ is a homeomorphism, because the inverse is given by $p \mapsto (p, g(p))$. This gives a chart at p_1. One can check that for any other choice $p_2, \ldots, \phi_2$, the transition function $\phi_2 \circ \phi_1^{-1}$ is C^∞. □

Definition 1.2.5. *A map $f : X \to Y$ between manifolds is C^∞ if*

1. *f is continuous*
2. *$\psi_j \circ f \circ \phi_i^{-1}$ is C^∞ for any charts $\phi_i : U_i \to V_i \subset \mathbb{R}^n$ and $\psi_j : U'_j \to V'_j \subset \mathbb{R}^m$.*

The last condition says that f is C^∞ when expressed in local coordinates. Using standard facts from calculus.

Theorem 1.2.6. *The composition of two C^∞.*

It follows that manifolds and C^∞ maps constitute a category. An isomorphism in this category is called a diffeomorphism. To be more explicit:

Definition 1.2.7. *A diffeomorphism between manifolds is a C^∞ bijection such that the inverse is also C^∞. Two manifolds are called diffeomorphic if a diffeomorphism exists between them.*

1.3 Riemann surfaces

We will postpone the discussion of definition of general complex manifolds, and focus on the important special case of Riemann surfaces (= complex curves) for now.

Definition 1.3.1. *A Riemann surface is a topological manifold X equipped with an altas $\phi_i : U_i \to V_i \subset \mathbb{C}$, such that $\phi_i \circ \phi_j^{-1}$ is holomorphic.*

By definition, a Riemann has complex local coordinate z in every chart. Coordinate changes are required to be holomorphic. A Riemann surface can be viewed as the C^∞ 2-manifold, with (real) coordinates $x = Re\, z, y = Im\, z$.

Example 1.3.2. *Any open subset of $\mathbb{C}$ is a Riemann surface.*

The next example is covered in a standard complex analysis class. It is the simplest example of Riemann surface which is not a subset of $\mathbb{C}$.

Example 1.3.3. *The Riemann sphere is $S^2 = \mathbb{C} \cup \{\infty\}$ has two charts $U_0 = \mathbb{C}$ with the identity $\phi_0 : \mathbb{C} \to \mathbb{C}$ or standard coordinate z, and $U_1 = \mathbb{C} - \{0\} \cup \{\infty\}$ with coordinate $\zeta = 1/z$. Alternatively, the sphere can be described as $\mathbb{P}^1_{\mathbb{C}}$, where $z = z_1/z_0$, and $\zeta = z_1/z_0$ in homogeneous coordinates.*

Example 1.3.4. *Given two $\mathbb{R}$-linearly independent complex numbers $a, b \in \mathbb{C}$, let $L = \mathbb{Z}a \oplus \mathbb{Z}b$ be the lattice generated by them. Let $E = \mathbb{C}/L$ with a projection $\pi : \mathbb{C} \to E$. If D is a disk of radius $r < \min(|a+b|, |a-b|)/2$ (so that D lies in a period parallegram), π will map D homeomorphically to its image $\pi(D)$. We take $\pi(D) \xrightarrow{\pi^{-1}} D$ to be a chart. In this way E becomes a Riemann surface called an* elliptic curve*. As a C^∞ manifold, $E = T^2$, and any two 2-tori are diffeomorphic. However, its structure as a Riemann surface depends in a subtle way on a, b.*

We will discuss many other examples later on. But let us consider a nonexample. As noted already, a Riemann surface is a C^∞ 2-manifold. The converse is not true however. For example, *a Moebius strip cannot be turned into a Riemann surface.* The reason is that the strip is not orientable. Informally, a surface in $\mathbb{R}^3$ is orientable if there is a nowhere nonzero normal vector field on it. With a little calculus, we can turn this into a better definition.

Definition 1.3.5. *A C^∞ 2-manifold is orientable if the Jacobian determinants of the coordinate changes*

$$\det \begin{pmatrix} \partial x'/\partial x & \partial x'/\partial y \\ \partial y'/\partial x & \partial y'/\partial y \end{pmatrix}$$

are either all strictly positive or strictly negative.

Theorem 1.3.6. *A Riemann surface is orientable.*

Proof. Let $z = x + iy, w = u + iv$ be holomorphic coordinates. The Cauchy-Riemann equations imply that

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} = u_x^2 + v_x^2 > 0$$

(In case, you forget what the Cauchy-Riemann says, remember that it means that the derivative is $\mathbb{C}$-linear, or equivalently that the Jacobian matrix commutes with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.) □

We can give another "proof" using the informal definition. It has the advantage of being calculation free. Suppose a surface $X \subset \mathbb{R}^3$ was a Riemann surface. Choose a unit tangent vector v at $p \in X$. Since the tangent plane can be identified with $\mathbb{C}$, we can multiply by i to get new tangent vector u. The cross product $v \times u$ gives a preferred unit normal. Note that the argument tells us that Riemann surfaces are not just orientable, but naturally oriented.

1.4 Complex manifolds

We start a few more remarks about holomorphic functions in one variable. Let us write

$$z = x + iy$$

as usual, and introduce complex valued differential forms

$$dz = dx + idy,\ d\bar{z} = dx - idy$$

(If you aren't sure what differential forms are, you can just be view them as formal expressions for now.) Therefore

$$dx = \frac{1}{2}(dz + d\bar{z})$$

$$dy = \frac{1}{2i}(dz - d\bar{z})$$

Given a C^∞ function $f : U \to \mathbb{C}$, the total differential

$$df = f_x dx + f_y dy = \frac{1}{2}(f_x - if_y)dz + \frac{1}{2}(f_x + if_y)d\bar{z}$$

We introduce operators

$$\partial f = \frac{1}{2}(f_x - if_y)dz$$

$$\bar{\partial} f = \frac{1}{2}(f_x + if_y)d\bar{z}$$

so that

$$d = \partial + \bar{\partial}$$

If we set $u = Re\, f, v = Im\, f$, then

$$\bar{\partial} f = \frac{1}{2}[(u_x - v_y) + i(u_y + v_x)]d\bar{z}$$

This makes it clear that the condition $\bar{\partial} f = 0$ is precisely the Cauchy-Riemann equations. Therefore

Lemma 1.4.1. *f is holomorphic iff $\bar{\partial} f = 0$.*

Let us now go to several variables.

Definition 1.4.2. *Let $U \subseteq \mathbb{C}^n$ be open. A function $f : U \to \mathbb{C}$ is holomorphic if it is C^∞ (or just C^1) and holomorphic in each variable, i.e. when all but one variable is fixed, $f(z_1, \ldots, z_n)$ is holomorphic in the remaining variable.*

We define the Cauchy-Riemann operator in several variables by

$$\bar{\partial} f = \sum_{j=0}^{n} \frac{1}{2}(f_{x_j} + i f_{y_j}) d\bar{z}_j$$

Theorem 1.4.3. *Given a C^∞ function $f : U \to \mathbb{C}$, the following are equivalent:*

1. *f is holomorphic.*
2. *$\bar{\partial} f = 0$*
3. *f is analytic: Writing $z = (z_1, \ldots, z_n)$, then for each $p \in U$, there exists an expansion*

   $$f(z+p) = \sum_{j_1, \ldots, j_n \geq 0} a_{j_1 \ldots j_n} z_1^{j_1} \ldots z_n^{j_n}$$

   which converges uniformly in a nbhd of 0.

We just give the briefest sketch. See Voisin or other references for more. The equivalence of 1 and 2 should be clear from what we said above. Also $3 \Rightarrow 1$ is clear. Let's consider the converse when $p = 0$. This requires a version of Cauchy's formula in several variables, which can be proved by induction:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1| = r_1} \cdots \int_{|\zeta_n| = r_n} \frac{f(\zeta)}{\prod(\zeta_j - z_j)} d\zeta_1 \wedge \ldots d\zeta_n$$

(ignore the $\wedge$ if you aren't sure what it means). The integrand can be expanded into a product of geometric series. Since this is uniformly convergent for small $|z|$, we can integrate term by term to obtain a power series expansion for $f(z)$.

We record another consequence of Cauchy's formula, which should be familiar from one variable.

Theorem 1.4.4 (Maximum principle)**.** *If f holomorphic function on a closed polydisk (= product of disks), then either $|f(z)|$ takes a maximum on the boundary or f is constant.*

In addition to analogues of results from one complex variable, there are also some new phenomena:

Theorem 1.4.5 (Hartogs theorem)**.** *Suppose that $n > 1$ and that $p \in U \subseteq \mathbb{C}^n$ is open. A holomorphic function on $U - \{p\}$ extends to U.*

A proof, along with stronger versions, can be found in any book on several complex variables. We are now ready to give the definition.

Definition 1.4.6. *A complex manifold of dimension n is topological manifold X equipped with charts $\phi_i : U_i \to V_i^* \subset \mathbb{C}^n$ such that $\phi_i \circ \phi_j^{-1}$ is holomorphic in the above sense.*

Note n above is the complex dimension. The real dimension would be $2n$. The axioms say about each point, we can find holomorphic (or analytic) coordinates $z_1, \ldots, z_n$. The real and imaginary parts $x_1 = Re\, z_1, y_1 = Im\, z_1, \ldots$ give $2n$ coordinates as the C^∞ manifold.

We already have many examples, such Riemann surfaces (where the complex dim =1), products of Riemann surfaces, and $\mathbb{P}^n_{\mathbb{C}}$. Here are a few examples, which come from algebraic geometry.

Example 1.4.7. *Let $f(z_1, \ldots, z_n)$ be a holomorphic function, e.g. a polynomial, such that the gradient $(\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ is nonzero along $X = f^{-1}(0) \subset \mathbb{C}^n$. Then X is a complex manifold of dimension $n-1$ called a nonsingular (algebraic or analytic) hypersurface of $\mathbb{C}^n$. This requires the holomorphic version of the implicit function theorem. We will say more about that later.*

Before explaining the projective analogue, note that a polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ in the homogenous coordinates does not define a function on $\mathbb{P}^n_{\mathbb{C}}$. However, if f is homogenous, then its zero set

$$V(f) = \{[a] \mid f(a) = 0\}$$

is well defined, because $f(a) = 0 \Rightarrow f(b) = 0$ for all $b \in [a]$. Also

$$f(z_0/z_i, \ldots, 1, \ldots z_n/z_n) \in \mathbb{C}[z_/z_0, \ldots, z_n/z_0]$$

gives a well defined non homogeneous polynomial in the true coordinates of U_i. The zero set of the latter, can be indentified with $V(f) \cap U_i$. Similar remarks apply to a set of homogenous polynomials.

Example 1.4.8. *Suppose that $f \in \mathbb{C}[z_0, \ldots, z_n]$ is a homogenous polynomial of degree d, such that the intersection of*

$$X = V(f)$$

with any U_i is a nonsingular hypersurface in $\mathbb{C}^n$, then X is a complex manifold in $\mathbb{P}^n_{\mathbb{C}}$. It is called a (projective algebraic) hypersurface of degree d.

In the last two examples, when X is zero set of a polynomial in $\mathbb{C}^2$ or $\mathbb{P}^2_{\mathbb{C}}$, it is called a nonsingular affine or projective algebraic plane curve. A degree one curve is isomorphic in an obvious to $\mathbb{P}^1$. The same is true when the degree is 2, but this is less obvious. For degree 3, we get something different, namely an elliptic curve. The proof uses the theory of elliptic functions.

In spite of the similarities between C^∞ manifolds and complex manifolds, there are also big differences. These stem from the fact that C^∞ functions are very flexible, while holomorphic functions are somewhat rigid. The following function on $\mathbb{R}^n$

$$f(x_1, \dots, x_n) = g(x_1)\dots g(x_n)$$

$$g(x) = \begin{cases} e^{1/(x-1)^2} e^{1/(x+1)^2} & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

is supported on the cube $[-1, 1]^n$. Viewing this as a function on a coordinate nbhd, and then extending by zero, shiows that any C^∞ manifold has many nonconstant global C^∞ functions. By contrast:

Theorem 1.4.9. *A holomorphic function on a connected compact complex manifold is constant.*

Proof. Let $f(z)$ be a holomorphic function on a compact manifold X. By compactness, $|f(z)|$ takes a maximum value at a point p_0. Then $S = \{p \in X \mid f(p) = f(p_0)\}$ is closed and nonempty. Given $p \in S$, and let U be coordinate nbhd of p equivalent to a polydisk. Then $f|_U$ is constant by the maximum principle. Therefore S is also open. Consequently $X = S$ by connectedness. □