INFINITE SERIES

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An infinite series is a sum

$$\sum_{n=0}^{\infty} c_n = c_0 + c_1 + \dots$$

where the c_i are complex numbers (and later on complex valued functions). This is said to converge to S if

$$\lim_{N\to\infty} S_N = S, \text{ where } S_N = \sum_{n=0}^{N} c_n$$

If there is no limit, the series is said to diverge. The basic example (discussed in class and the book) is

THEOREM 1. *If $|c| < 1$, then the geometric series*

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$$

converges.

The comparison test allows us to construct other examples from this:

THEOREM 2 (Comparison Test). *If $|c_n| \le M_n$ for all n and if*

$$\sum_{n=0}^{\infty} M_n$$

converges, then

$$\sum_{n=0}^{\infty} c_n$$

converges.

EXAMPLE 1.

$$\frac{1}{n!} = \frac{1}{1\cdot 2\cdot 3\cdots n} \le \frac{1}{2\cdots 2} = \frac{1}{2^{n-1}}$$

Therefore $\sum_{n=0}^{\infty} i^n/n!$ converges by the comparison test.

Suppose that a_n is a sequence of real numbers with limit $a = \lim a_n$, then as n and m get large, a_n and a_m get closer to a, therefore closer to each other. More precisely for every $\epsilon > 0$, there exists N such that $|a_n - a_m| < \epsilon$ for $n, m > N$. A sequence with this property is called a Cauchy sequence To prove that the comparison test works, we need the following fact which is an *axiom*[1] about the real numbers system, rather than a theorem.

[1] Although in some treatments of real analysis, this is sometimes deduced from another axiom called the least upper bound axiom.

AXIOM 1 (Completeness of the real numbers)**.** *Suppose a_n is Cauchy sequence of real numbers, then $\lim a_n$ exists as a real number.*

To apply this to complex sequences, we note that

$$\lim_{n\to\infty}(a_n + ib_n) = \lim_{n\to\infty} a_n + i \lim_{n\to\infty} b_n$$

if both limits exist on the right.

THEOREM 3. *Suppose that c_n is a sequence of complex numbers such that for every $\epsilon > 0$, there is an N so that the"tail" $|\sum_{j=n}^m c_j| < \epsilon$ for $m > n > N$. Then $\sum_{j=0}^\infty c_j$ converges.*

Proof. Let $c_j = a_j + ib_j$, and let $A_N = \sum_{j=0}^N a_j$ and $B_N = \sum_{j=0}^N b_j$. We see that $\epsilon > 0$, there is an N so that the $|A_m - A_n| \le \sum_{j=n}^m c_j| < \epsilon$ for $m > n > N$. So the completeness axiom shows that $\sum_{j=0}^N a_j = \lim_N A_N$ exists. The same argument shows that $\sum_{j=0}^N b_j = \lim_N B_N$ exists. So $\sum_{j=0}^\infty c_j = \lim A_N + i \lim B_N$. □

Proof of theorem 2. By the triangle inequality together with the hypothesis, we have

$$|\sum_n^m c_j| \le \sum_n^m |c_j| \le \sum_n^m M_n$$

The sequence $S_n = \sum_0^n M_n$ converges so its Cauchy, that $\sum_n^m M_n$ can be made as small as possible for $m > n > N$, as $N \to \infty$.

□

THEOREM 4. *Suppose that for some real $0 < b < 1$ and N we have $c_n \neq 0$ and $|c_{n+1}/c_n| \le b$ for $n \ge N$, then $\sum_{n=0}^\infty c_n$ converges.*

Proof. If $n \ge N$, then

$$|c_n| \le b|c_{n-1}| \le b^2|c_{n-2}| \le \dots b^{N-n}|c_N|$$

By the comparison test

$$\sum_{n=0}^\infty c_n = \sum_{n=0}^{N-1} c_n + c_N \sum \frac{c_n}{c_N}$$

converges. □

COROLLARY 1 (The ratio test)**.** *If*

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = a$$

exists and $a < 1$, then $\sum_{n=0}^\infty c_n$ converges.

Proof. Pick $a < b < 1$ (e.g. $b = (1 + a)/2$), then for all but finitely many n, $|c_{n+1}/c_n| \le b$ □

We can use this to improve an earlier example.

EXAMPLE 2. *For any complex number a consider $c_n = a^n/n!$. Then*

$$\lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = \lim \frac{|a|}{n} = 0$$

so $\sum a^n/n!$ converges.