Perverse sheaves on the disk

Fix a small disk Δ around 0. ("Small" just means that singularities in the discussions below occur only at 0.) Let z be a coordinate, and $\Delta^* = \Delta - \{0\}$ with inclusion j.

Recall that an object F in $D^b_c(\Delta, \mathbb{Q})$ is a **perverse sheaf** if

$$\dim \mathit{supp}\, \mathcal{H}^i(F) \leq -i,$$

and the same holds for the Verdier dual $DF = \mathbb{R}\mathcal{H}om(F, \mathbb{Q}[2])$.

$Perv(\Delta)$

Fix a perverse sheaf F on Δ. We know (or can assume w.l.o.g.) the following facts.

- P1 $\mathcal{H}^i(F) = 0$ for $i \neq -1, 0$.
- P2 $\mathcal{H}^0(F)$ is supported at 0.
- P3 $\mathcal{H}^{-1}(F)$ has no sections supported at 0.
- P4 $\psi = \mathcal{H}^{-1}(F)|_{\Delta^*}$ is a local system.

Proof of P3 (which wasn't discussed earlier): Since $\mathcal{H}^1(DF) = 0$, Verdier duality implies local cohomology $H_0^{-1}(F) = 0$. This forces $H_0^0(\mathcal{H}^{-1}(F)) = 0$. Therefore $H^0(\Delta^*, \mathcal{H}^{-1}(F)) \to H^0(\Delta, \mathcal{H}^{-1}(F))$ is injective.

Let $Perv(\Delta)$ denote the category of objects satisfying above conditions.

Perv(Δ) continued

The first goal is to give an elementary description of this category.

ψ can (and will) be viewed as a $\mathbb{Q}[T, T^{-1}]$-module, with T given by the monodromy.

Proposition

There is an essentially surjective functor from *Perv*(Δ) to the category $\mathcal{T}$ of triples

$$(\psi, \alpha^{-1} : V^{-1} \to \ker[\psi \stackrel{T-1}{\to} \psi], \alpha^0 : V^0 \to \mathrm{coker}[\psi \stackrel{T-1}{\to} \psi])$$

where ψ is a $\mathbb{Q}[T, T^{-1}]$-module, which is finite dimensional over $\mathbb{Q}$, α^i are linear maps from finite dim vector spaces, such that α^{-1} is injective.

(I think it's an equivalence but didn't check carefully.)

Sketch of Proof

The functor from $Perv(\Delta) \to \mathcal{T}$ is given as follows. Given a perverse sheaf F, ψ is given as above, $V^i = \mathcal{H}^i(F|_0)$. The maps α^i are induced by the adjunction $F \to (\mathbb{R}j_*j^*F)$. The first map α^{-1} is injective by P3.

Given an object $(\psi, \alpha^{-1}, \alpha^0)$ in $\mathcal{T}$, we can find a complex of vector spaces $W^\bullet$, and a morphism $\alpha : W^\bullet \to \mathbb{R}j_*\psi[1]|_0$ such that $\alpha^i = \mathcal{H}(\alpha)$. Let F fit into the distinguished triangle

$$F \to W \oplus Rj_*\psi[1] \to Rj_*\psi[1]|_0$$

One can see that $F \in Perv(\Delta)$ and that it maps to the given object.

Verdier's theorem (set up)

Given an object of $\mathcal{T}$, set

$$\phi = (\psi / \operatorname{im} \alpha^{-1}) \oplus V^0$$

Projection to the first factor gives the (canonical) map

$$can : \psi \to \phi$$

We have a map $T - 1 : \psi \to \psi$. Since V^{-1} lies in the kernel of this, $T - 1$ factors through ϕ. So we have the (variation) map

$$var : \phi \to \psi$$

such that the composite

$$\psi \overset{can}{\to} \phi \overset{var}{\to} \psi$$

is $T - 1$

Verdier's theorem

Theorem (Verdier)

Perv(Δ) *is equivalent to the category* Q *of quivers* $(\psi, \phi, can : \psi \to \phi, var : \phi \to \psi)$*, such that* $var \circ can + 1$ *is an automorphism of* ψ*.*

Ref: Verdier, *Extension of a perverse sheaf over a closed subspace.*

Corollary

Given a local system ψ *on* Δ^**, its extensions to Perv*(Δ) *are parameterized by factorizations* $\psi \to \phi \to \psi$ *of* $T - 1$*.*

Extending local systems

Given a local system ψ on Δ^*, we list the following extensions to $Perv(\Delta)$ in regular sheaf notation, BBD (Beilinson-Bernstein-Deligne) notation, along with characterizations of the corresponding quivers:

1. $\mathbb{R}j_*\psi[1]$ ($= {}^p j_*\psi[1]$ in BBD) corresponds to the quiver $(\psi, \psi, can = T - 1, var = id)$ isomorphism.
2. $j_!\psi[1]$ ($= {}^p j_!\psi[1]$ in BBD) corresponds to the quiver $(\psi, \psi, can = id, var = T - 1)$.
3. the minimal extension $j_*\psi[1]$ ($= {}^p j_{!*}\psi[1]$ in BBD) corresponds to the quiver

$$\mathrm{im}(\psi, \psi, can = id, var = T - 1) \to (\psi, \psi, can = T - 1, var = id)$$

under the morphism $(id, T - 1)$.

Vanishing cycles

The construction I gave for ϕ above was totally ad hoc. So it may be good to explain what it really means. The name of the game is vanishing cycles [SGA7]. Since this will be discussed later in the seminar, I will be sketchy.

Let $\tilde{\Delta}^* \to \Delta$ be the universal cover, and let $\tilde{\Delta} = \tilde{\Delta}^* \cup \{0\}$ as a set with inclusion $\tilde{j} : \tilde{\Delta}^* \to \tilde{\Delta}$. We can give this a topology so that the map $p : \tilde{\Delta} \to \Delta$ looks like an infinite sheeted branched cover ramified at 0.

Nearby cycles

Given $F \in D^b(\Delta)$, let

$$\Psi F = (\mathbb{R}\tilde{j}_*\tilde{j}^* p^* F)|_0$$

be the "nearby cycles" functor.

The terminology is suggestive. When $f : X \to \Delta$ is a family of complex projective varieties and $F = \mathbb{R}f_*\mathbb{Z}$, ΨF is a complex of sheaves at 0 which computes the cohomology of the nearby fibres $X_t, t \neq 0$.

Vanishing cycles

Given $F \in D^b(\Delta)$, we have a natural map

$$F|_0 \to \Psi F$$

induced by adjunction. We can complete this to a distinguished triangle

$$F|_0 \to \Psi F \to \Phi F \to F|_0[1]$$

The middle term in red is called the "sheaf" of vanishing cycles.

This measures the difference between $H^*(X_t)$ and $H^*(X_0)$ in the geometric example.

The following is not hard.

Lemma

When F is perverse,

$$\Psi F = \psi[1]$$

$$\Phi F = \phi[1]$$

where ψ and ϕ were constructed earlier.

The corresponding statement in higher dimensions is that $\Psi F[-1]$ and $\Phi F[-1]$ are perverse if F is. This is a nontrivial theorem due to Gabber.

Ref. Brylinkski, *Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques*

Key Insight

In Saito's theory of (mixed) Hodge modules, Gabber's theorem is turned into a definition!

Regular holonomic D-modules on the disk

From Verdier's theorem, we obtain

Corollary

The category of regular holonomic D-modules on a "small" disk Δ is equivalent the category $Q = \{(\psi, \phi, can, var)\}$ above.

It's good to understand things more explicitly. Let us start with a (finite rank) a vector bundle V with a connection ∇. This corresponds to a local system on Δ^*.

Can we describe some (or all) the regular holonomic D_Δ-modules extending (V, ∇)?

We note that $j_* V$ is a D_Δ-module, but it is not quasi-coherent, so it is not admissible for our purposes.

Regularity means that we can find an extension of (V, ∇) to a bundle $\bar{V}$ with logarithmic connection

$$\bar{\nabla} : \bar{V} \to \Omega^1_\Delta(\log 0) \otimes \bar{V}$$

The extension is not unique, and it is useful to understand this. $\bar{V}$ is necessarily trivial. If we choose a basis for it, then $\bar{\nabla}$ can be represented by a matrix of 1-forms

$$A = R\frac{dz}{z} + \text{holomorphic part}$$

R is called the residue.

Proposition (Deligne)

For each interval $[r, r+1)$ *(resp.* $(r, r+1]$*), there exists unique extension* $V^{\geq r}$ *(resp.* $V^{>r}$ *)* $\subset j_* V$ *such that the eigenvalues of the residue* R *lie in that interval.*

Ref. Deligne, *Équations différentielles à points singuliers réguliers*

Note that $V^{>r}$ is generally not a D-module. However, $\partial V^{>r} \subseteq V^{>r-1}$. Therefore we get at least one extension of the desired type.

Lemma

$M = \bigcup_r V^{>r}$ *is a (quasicoherent)* D_Δ*-module extending* V*.*

This typically won't be the minimal extension, but it does contain it. Notice that M comes with a filtration $V^\bullet$, which is essentially the Kashiwara-Malgrange filtration discussed later.