Hodge theoretic background

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Hodge structures

From the Hodge decomposition for smooth projective varieties (or more generally compact Kähler manifolds), one extracts the following notion.

A (integral) Hodge structure

of weight n consists of a finite $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ (or lattice $H_{\mathbb{Z}}$) and a decomposition

$$H = H_{\mathbb{Q}} \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{pq}$$

such that $\overline{H}^{pq} = H^{qp}$

In addition to geometric examples, it is easy to construct artificial examples by hand.

The category of Hodge structures

A morphism of Hodge structures

$f : H \to G$ is a $\mathbb{Q}$-linear map preserving the bigrading,

Remarks:

1. Up to isomorphism, there is a unique one dimensional Hodge structure of weight $2n$ denoted by $\mathbb{Q}(-n)$.
2. A map between Hodge structures of different weights is 0.
3. The category of Hodge structures is abelian.
4. The Hodge filtration is defined by

$$F^p H = H^{p,n-p} \oplus H^{p+1,n-p-1} \oplus \dots$$

This determines the bigrading. Morphisms can equivalently defined as maps preserving the filtration.

Given two Hodge structures H and G of weight n and m, the vector space tensor product $H \otimes G$ carries a natural Hodge structure of weight $n+m$. Many deeper results about Hodge structures require:

A polarization

on a Hodge structure of weight n is a morphism

$$Q : H \otimes H \to \mathbb{Q}(-n)$$

satisfying the Hodge-Riemann relations. Explicitly, if C acts by i^{p-q} on H^{pq}, then $i^n Q(-, C-)$ is positive definite symmetric.

Ex: The nth cohomology of projective manifold carries a polarization. When $n = 1$, it is

$$Q(\alpha, \beta) = \int_X c_1(\mathcal{O}(1))^{\dim X - 1} \wedge \alpha \wedge \beta$$

Period domains

Theorem (Griffiths)

Fix a lattice $H_{\mathbb{Z}}$, a quadratic form Q on it, and a collection of natural numbers $h^{p,n-p}$ whose sum is rank $H_{\mathbb{Z}}$. Then set of Hodge structures of weight n on $H_{\mathbb{Z}}$ polarized by Q with given Hodge numbers forms a homogeneous complex manifold D, called the period domain.

By construction D is an open subset of a flag variety $\check{D}$ called the compact dual.

Example

When $H_{\mathbb{Z}} = \mathbb{Z}^2$ with $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $h^{10} = h^{01} = 1$, $\check{D} = \mathbb{P}^1$, and D is the unit disk.

Smooth families

Suppose that $f : X \to Y$ is a family of smooth projective varieties over a smooth base. Then

- f is topologically a fibre bundle, therefore the direct images $L_i = R^i f_* \mathbb{Z}$ are local systems.
- Consequently
$$V_i = \mathcal{O}_Y \otimes L_i$$
becomes a vector bundles withan integrable connection ∇, called the Gauss-Manin connection.
- This can be identified with relative de Rham cohomology
$$V_i = \mathbb{R}^i f_* \Omega^\bullet_{X/Y}$$
- Define the decreasing filtration
$$F^p V_i = \operatorname{im} \mathbb{R}^i \Omega^{\geq p}_{X/Y} \to V_i$$

Theorem

1. *(Strictness) The map* $\mathbb{R}^i\Omega^{\geq p}_{X/Y} \to V_i$ *is injective for each* p*.*
2. $F^pV_i \subset V_i$ *are subbundles, called Hodge bundles.*
3. *(Griffiths transversality)* $\nabla(F^p) \subset \Omega^1_Y \otimes F^{p-1}$

Remarks:

1. Strictness of the filtration might seem merely technical, but it is essential for the rest of the theorem, and many other things. The only proofs are via ordinary or p-adic Hodge theory.
2. If we define $F_pV = F^{-p}V$, then Griffiths transversality just says that $F_\bullet$ is a good filtration on the D-module V_i.
3. One gets an equivariant holomorphic maps from the universal cover $\tilde{Y}$ to period domain D. Griffiths transversality does in fact say this period map is transverse to a certain fibration.

VHS

Abstracting the previous example.

A variation of Hodge structures

of weight n is a collection $(L, V, \nabla, F^\bullet V)$ consisting of a local system etc. satisfying the conditions of the previous slide, such that, in addition, $F^\bullet V$ induces a Hodge structure of weight n on the fibres. A polarization of VHS is a flat map $Q : L \otimes L \to \mathbb{Z}$ polarizing the fibres.

The defintion goes back to Griffiths in the late 1960's. 20 years later, Saito took it as the prototype of a Hodge module. For the general notion, L is replaced by a perverse sheaf, and (V, ∇, F) by a filtered D-module.

VHS on a punctured disk

Let $0 \in \Delta$ be a disk with coordinate $z = re^{i\theta}$, and $\Delta^* = \Delta - \{0\}$ Let $(L, \dots)$ be a polarized VHS on Δ^*.

Theorem (Borel)

The monodromy transformation of L is quasi-unipotent, i.e. the eigenvalues are roots of unity.

Ref. Schmid, *Variations of Hodge structure...*, Inventiones 1973

In the geometric setting, this was independently due to Brieskorn, Grothendieck, Landman...

In view of the last theorem, after passing to an unramified cover we can assume that the monodromy T is unipotent.

Theorem (Schmid)

In the unipotent case, there is a good asymptotic description of the period map.

I purposely kept this vague. However, I would like to spell out one specific result needed later. The bundle V, viewed as a C^∞ vector bundle, carries a Hermitian metric $|| - ||_{Hodge}$ called the Hodge metric induced by the polarization. Schmid gives precise estimates on the norm of sections of V in terms of monodromy.

$N = \log T$ is nilpotent. Assume, for simplicity, that $N^2 = 0$. Choose $t \in \Delta^*$, then we have filtration

$$M_{-1} = \operatorname{im} N \subseteq M_0 = \ker N \subseteq M_1 = V_t$$

called the monodromy (weight) filtration. More generally, $M_\bullet$ exists for any nilpotent N such that $NM_i \subseteq M_{i-2}$ and a hard Lefschetz property holds.

Theorem (Schmid)

A nonzero element v of M_i viewed as a multivalued section of the Deligne extension $\bar{V} = V^{\geq 0}$ satisfies

$$||v||_{Hodge} = O(|\log r|^{i/2})$$

Zucker's theorem

Let X be compact Riemann surface, $S \subset X$ a finite set, and $U = X - S$ with inclusion $j : U \to X$. Let L be a local system on U. Define the intersection cohomology

$$IH^i(X, L) = H^i(X, j_* L)$$

(Recall $j_* L[1]$ is the minimal extension of the perverse sheaf $L[1]$. This is also called the intersection cohomology complex $IC(L)$, up to shift depending on who you ask.)

Fix a polarized VHS $(L, V, \ldots)$ of weight n on U.

Theorem (Zucker)

$IH^i(X, L)$ carries a natural Hodge structure of weight $i + n$.

Ref. Zucker, *Hodge theory with degenerating coefficients...*, Annals 1979.
Remarks.

1. Saito uses this result in an essential way to prove the category of Hodge modules is stable under direct images.
2. One can view this as a Hodge theoretic analogue of Deligne's purity theorem (thm 2, *La conjecture de Weil II*).
3. When X is a modular curve, and L comes from the m symmetric power of the standard representation of $SL_2(\mathbb{R})$, the bigraded components $IH^1(X, L)$ are the modular forms of weight $m + 2$ or their conjugates. This was first observed – in different language – by Shimura.

The way I stated Zucker's theorem is too imprecise to be useful. Let me describe the the story in more detail.

What he shows is that there is a subcomplex $\Omega^\bullet_{L^2}$ of

$$V^{\geq 0} \to \Omega^1_X(\log S) \otimes V^{\geq 0}$$

with an isomorphism

$$IH^i(X, L) = \mathbb{H}^i(\Omega^\bullet_{L^2})$$

Furthermore, the Hodge filtration is induced from a filtration on $\Omega^\bullet_{L^2}$.

To see where this comes from, we need to realize this by harmonic forms.

Assume without too much loss of generality, that U is hyperbolic. This together with the Hodge metric determines and inner product on a space of V-valued forms $\mathcal{E}^\bullet_{L^2}(X)$. Let ∇^* be the adjoint to the connection, and define the Laplacian $\Delta = \nabla\nabla^* + \nabla^*\nabla$ as usual.

Theorem

$IH^i(X, L) \otimes \mathbb{C}$ is isomorphic to the space of harmonic forms in $\mathcal{E}^i_{L^2}(X)$.

This comes down to usual Hodge theorem for elliptic operators, plus

Theorem

1. *(Poincaré lemma) $j_*L \otimes \mathbb{C}$ is resolved by the complex of sheaves $\mathcal{E}^\bullet_{L^2}$ of locally L^2 C^∞-forms.*
2. *(Dolbeault lemma) $j_*L \otimes \mathbb{C}$ is resolved by the complex of sheaves $\Omega^\bullet_{L^2}$ of locally L^2 holomorphic forms.*

Sketch of proof of Dolbeault.

It suffices to check exactness at each $0 \in S$. So we may replace X by a disk of radius ϵ centered at 0, with local parameter z.
There are 3 steps.

1. Reduce to the unipotent case.
2. Identify $\Omega^\bullet_{L^2}$ with
$$0 \to j_* L_{\mathbb{C}} \to (M_0 + z\bar{V}) \to \frac{dz}{z}(M_{-2} + z\bar{V}) \to 0$$
where $\bar{V} = V^{\geq 0}$ is the Deligne extension, and $M_\bullet$ is the monodromy filtration.
3. Show that this complex is exact. This follows from the formal properties of $M_\bullet$.

□

Proof Cont.

I just explain step 2. Given a section v of V, with $v \in M_k - M_{k-1}$, by Schmid's norm estimates given earlier,

$$||z^n v||_{Hodge} \sim r^{2n}|\log r|^k$$

Therefore

$$||z^n v||^2 \sim \int_0^{2\pi}\int_0^{\epsilon} \underbrace{r^{2n}|\log r|^k}_{||z^n v||^2_{Hodge}} \underbrace{\frac{dr d\theta}{r|\log r|^2}}_{\text{Poincaré vol}} < \infty$$

iff $k \leq 0, n \geq 0$, or $k > 0, n > 0$. This implies the space of L^2 sections of V is isomorphic to $M_0 + z\bar{V}$ as claimed.

$\square$

Proof Cont.

A similar estimate holds

$$||z^n \frac{dz}{z} v||_{Hodge} \sim r^{2n} |\log r|^{k-2}$$

proves the result for differentials.

Given a local section of v of $j_* L_{\mathbb{C}}$, it is invariant under monodromy. So it lies M_0.

$\square$

Corollary (of proof)

The complex $\Omega^\bullet_{L^2}$ has an algebraic description independent of the metric.

We define a bigrading on $\mathcal{E}^\bullet_{L^2}(X)$ by total bidegree, i.e. a (p,q) form with coefficients in $V^{p'q'}$ has bidgree $(p+p', q+q')$. The space of harmonic forms inherits this bigrading by a version of the Kähler identities.
Therefore we get a Hodge structure

In more detail, V has a second connection ∇_{metric} such that it preserves the Hodge metric and $\nabla^{01}_{metric} = \bar{\partial}$.
The difference can be written as

$$\nabla - \nabla_{metric} = \theta + \bar{\theta}$$

where θ is a holomorphic section of $\Omega^1_U \otimes V$ called (by Simpson) the Higgs field.

Define a new Laplacian by

$$\Box = (\bar{\partial} + \theta)(\bar{\partial} + \theta)^* + (\bar{\partial} + \theta)^*(\bar{\partial} + \theta)$$

One has a generalization of the classical Kähler identities

Proposition

$\Delta = 2\Box$.

It follows that a form in $\mathcal{E}_{L^2}$ is harmonic $\Leftrightarrow$ its bigraded components are harmonic $\Leftrightarrow$ its conjugate is harmonic. Therefore we get a Hodge structure.