Applications Hodge Modules

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Viehweg's conjecture

I want to explain a part of the following paper:

[PS] Popa, Schnell, Viehweg's hyperbolicity conjecture for families with maximal variation, Inventiones 2017

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Their main result is

Theorem 1 (PS)

Let $f : Y \to X$ be surjective morphism of smooth projective varieties with connected fibres. Suppose $D \subset X$ is divisor such that f is smooth over $X - D$. Suppose that f has maximal variation and the smooth fibres have general type, then (X, D) has log general type, i.e. $\omega_X(D)$ is big***

*The image of the map of $X - D$ to moduli of fibres is big as possible.
** A line bundle L is big if $h^0(L^{\otimes k}) \sim Ck^{\dim X}$ (the fastest possible rate).

Variation of Hodge structures

The idea of applying variations of Hodge structures to geometric problems is not new, but the novelty of [PS] is the use of Hodge modules. I want to mention an earlier result in this direction.

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Theorem 2 (Zuo, 2000)

If $D \subset X$ is an snc divisor, such that $X - D$ carries a polarized VHS for which the period map is injective somewhere, then (X, D) is log general type.

The proof hinges on a positivity result that will be explained later.

Background on Hodge modules

Recall that a Hodge module consists of a perverse sheaf plus a filtered D-module M satisfying a bunch of conditions. I'll always work with a left D-module M, and I'll conflate it with the Hodge module.

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Recall also the main example is given as follows. Given a polarizable variation of Hodge structure M^o on a Zariski open $U \subset X$, there is a unique way to extend it a Hodge module M on X, with no nonzero factors on $X - U$. One says that M has strict support on X. The underlying D_X-module of M is just the minimal extension of the the D_U-module M^o. The Hodge filtration F is more complicated and given by Saito's formula involving F and V discussed earlier. Recall also that if the VHS has weight k, then the Hodge module will have weight $k + \dim X$.

Let $f : Y \to X$ be surjective map of smooth projective varieties, and let $r = \dim Y - \dim X$ be the relative dimension. Let M is the minimal extension of the VHS associated to the rth cohomology $R^r f_* \mathbb{Q}|_{X-\Delta}$, where $\Delta \subset X$ is the discriminant.

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Proposition 1

$$Gr_k^F M = \begin{cases} 0 & k < -r \\ f_* \omega_{Y/X} & k = -r \end{cases}$$

Sketch.

Let $\mathbb{Q}_Y$ be regarded as a VHS of weight 0, and $\mathbb{Q}_Y^H$ denote the corresponding Hodge module . Then $M' = \mathcal{H}^d f_* \mathbb{Q}_Y^H$, where $d = \dim Y$, and $\mathbb{Q}_Y^H$. By Saito, M' is a sum $M \oplus M''$, where M has strict support X, and M'' is supported on the discriminant Δ. Although the statement is about M, it suffices to prove it for M' because $f_*\omega_{Y/X}$ is torsion free, and any contribution from M'' would be torsion.

□

Sketch Cont.

The filtered D-module associated to M' is the direct image of $(\mathcal{O}_Y, F_\bullet \mathcal{O}_Y)$, where F is the trivial filtration, in the filtered derived category. Combined with various other results of Saito, we obtain that

$$Gr_k^F M' = \mathbb{R}f_*(\omega_{Y/X} \otimes_{\mathcal{O}_Y} Gr^F_{k+r}\mathcal{O}_Y \otimes^{\mathbb{L}}_{S_Y} f^* S_X)$$

where $S_X = S^*\mathcal{T}_X$, and furthermore the complex on right is splits or is formal in the sense it is a sum of its cohomology. By computing the cohomology of this complex by a Koszul complex one can verify the proposition.

□

Proof of theorem 1

When X is nonuniruled, theorem 1 is reduced to theorems 3 and 4 below. First some terminology. Say that M is large wrt a divisor $D \subset X$ if

1. D contains "singularities" of M, i.e. it's a VHS away from D.
2. There exists a big line bundle A such that

$$A \subset F_p M \otimes \mathcal{O}(\ell D)$$

where $\ell \geq 0$, and p is the minimal index for which $F_p M$ is nonzero.

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It will be necessary to extend the notion to where one can speak of largeness of a graded submodule $G \subseteq Gr^F(M)$. I'll say (M, G) is large,

Theorem 3

Let $f : Y \to X$ be a surjective map with connected fibres between smooth projective varieties, with discriminant divisor $D \subset X$. Assume that $\det f_*\omega_{Y/X}^{\otimes m}$ *is big for some* $m > 0$. *Then there exists a Hodge module* M, *and a graded submodule* $G \subseteq Gr^F(M)$ *such that* (M, G) *is large wrt* D.

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Theorem 4

Let X be a smooth nonuniruled projective variety. Assume there exists (M, G) as above which is large wrt D. Then $\omega_X(D)$ is big.

Sketch of proof of theorem 3.

After a series of geometric reductions, one can assume that the mth power of

$$B = \omega_{Y/X} \otimes f^* L^{-1}$$

has a nonzero section s, where $L = A(-\ell D)$, with A an ample line bundle, and $m > 0, \ell \geq 0$.

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Let $Y' \to X$ be the m-fold cyclic cover branched over $s = 0$. Let Z be a desingularization of Y'. Let $h : Z \to X$ be the obvious composition, and h^o the smooth part. Let M be the minimal extension of $R^{\dim Z} h^o_* \mathbb{Q}$.

□

Sketch Cont.

Prop 1 shows that the minimal $p = -r$ and

$$Gr_p(M) = h_* \omega_{Z/X}$$

Combined with the previous assumptions, we see that when $m = 1$, in which case we can take $Z = Y$, this implies that M is large.

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The general case, involves an extra step to choose a $G \subset Gr^F(M)$ whose lowest piece satisfies $G_p = f_*\omega_{Y/X}$ .Then (M, G) is large.

□

Weak positivity

The proof of theorem 4 hinges on certain positivity results. A divisor is pseudo-effective if it lies in the closure of the cone of effective divisors. Viehweg has extended the notions of bigness and pseudo-effectivity from divisors to torsion free coherent sheaves. The latter is called weak positivity.

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Lemma 1

Let E and F be divisors.

1. *A quotient of a weakly positive sheaf is weakly positive.*
2. *If $\mathcal{O}(E)$ contains a weakly positive sheaf, then E is pseudo-effective.*
3. *The tensor product of a big line bundle with a weakly positive sheaf is big. In particular, if E is pseudo-effective, and F is big, then $E + F$ is big.*

Weak positivity

Let M be a Hodge module on X (resp. a polarized VHS on $X - D$ with unipotent monodromy around D, which is an snc divisor). We have an associated filtered D-module $(M, F_\bullet M)$. This gives maps

$$\theta_k : Gr_k^F M \to Gr_{k+1}^F \otimes \Omega_X^1$$

or

$$\theta_k : Gr_k^F M \to Gr_{k+1}^F \otimes \Omega_X^1(\log D)$$

Let $K_k(M)$ denote the kernel in either case.

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Zuo's theorem stated earlier was refined by Brunebarbe (2018) to show that with the same assumptions $\Omega_X^1(\log D)$ is big. These results were deduced with the help of the first half of the next theorem.

Theorem 5

1. *(Zuo) If M is a polarized VHS on $X - D$ (D an snc divisor), then the dual $K_k(M)^\vee$ is weakly positive for any k.*
2. *(Popa-Wu) The same holds for any Hodge module with strict support X.*

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Part 2 is reduced to part 1. The polarization gives a Hodge metric on $K_k(M)^\vee$ with singularities along D. Zuo proves 1 by showing the curvature of the Hodge metric is nonnegative. By a theorem of Kollár, the singularities are mild enough that Chern-Weil still works. (Brunebarbe gave a different proof of a strengthened form of part 1.)

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Sketch.

Assume for simplicity that M is large. Then we have a big line bundle A and $\ell \geq 0$ with an inclusion

$$A(-\ell D) \to F_p M = Gr_p^F M$$

with p minimal. We can compose with θ_p to get a map

$$(*) \quad A(-\ell D) \to Gr_{p+1}^F M \otimes \Omega_X^1$$

□

Sketch Cont.

We consider two cases where (*) is zero, or not zero.

(1) Start with the first case. Then we get an injection

$$A(-\ell D) \to K_p(M)$$

Since $A(-(\ell+1)D) \subset A(\ell D)$, we can assume wlog that $\ell > 0$, and in fact, bigger than any constant. Dualizing gives a nonzero map

$$K_p(M)^\vee \to A^{-1}(\ell D)$$

Since the sheaf on the left is weakly positive, this forces $A^{-1}(\ell D)$ to be pseudo-effective by the lemma. Since A is big, ℓD, and therefore D, is big by the lemma. Since X is not uniruled, a theorem of Bouksom-Demailly-Paun-Peternell implies that ω_X is pseudo-effective. Therefore $\omega_X(D)$ is big.

□

Sketch Cont.

(2) The remaining case is where (*)

$$A(-\ell D) \to Gr^F_{p+1} M \otimes \Omega^1_X$$

is nonzero. The strategy is broadly similar. Composing the above map with successive maps in the chain

$$Gr^F_{p+1} M \otimes \Omega^1_X \overset{\theta_{p+1}}{\to} Gr^F_{p+2} M \otimes (\Omega^1_X)^{\otimes 2} \overset{\theta_{p+2}}{\to}$$

we eventually get 0 (because the $A(-\ell D)$ is locally free but the sheaves above are eventually 0 on U). This results in an injection

$$A(-\ell D) \to K_{p+s}(M) \otimes (\Omega^1_X)^{\otimes s}$$

for some s. □

Sketch Cont.

Dualizing, and using weak positivity of K_{p+s} and bigness of A results in a big subsheaf of

$$(\Omega^1_X)^{\otimes s}(\ell D)$$

One can deduce from this (with some work) that its top exterior power

$$\det[(\Omega^1_X)^{\otimes s}(\ell D)] = \omega_X^{\otimes s}(kD), \quad k = \ell s \dim X,$$

is big. Therefore

$$(\omega_X(D))^{\otimes k} = \underbrace{\omega_X^{\otimes k-s}}_{\text{psd. eff.}} \otimes \underbrace{\omega_X^{\otimes s}(kD)}_{\text{big}}$$

is big. So $\omega_X(D)$ is big. □