Perverse t-structures

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Recall: triangulated categories

Definition

A *triangulated category* is an additive category C together with

- an autoequivalence $+1 : \mathsf{C} \to \mathsf{C}$ (*shift functor*), and
- a class of *distingushed triangles* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$, satisfying

(TR1) d.t.'s are closed under isomorphisms, and each triangle $X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$ is a d.t.,

(TR2) every $X \xrightarrow{f} Y$ can be completed to a d.t. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$,

(TR3) a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a d.t. if and only if $Y \xrightarrow{-g} Z \xrightarrow{-h} X[1] \xrightarrow{-f[1]} Y[1]$ is a d.t.,

Definition (continued)

(TR4) Given a commutative diagram of solid arrows

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

with d.t.'s as rows and the solid square commutative, there exists the dashed arrow making everything commutative.

(TR5) In any commutative diagram of solid arrows

$$\begin{tikzcd}
X \arrow[dr] \arrow[rr, bend left] & & Z \arrow[dr] \arrow[rr, bend left] & & C \arrow[dr] \arrow[rr, dashed, bend left] & & A[1] & \cdots \\
& Y \arrow[ur] \arrow[dr] & & B \arrow[ur, dashed] \arrow[dr] & & Y[1] \arrow[ur] & & \cdots \\
& & A \arrow[ur, dashed] \arrow[rr, bend right] & & X[1] \arrow[ur] & & & \cdots
\end{tikzcd}$$

where all squiggles are d.t.'s, the dashed arrows exist, make everything commutative and form a d.t.

Recall: triangulated categories

Example

A a Grothendieck category, then D $=$ D(A) or D^b(A) is a triangulated category, where d.t.'s are triangles isomorpic to

$$X \xrightarrow{f} Y \to Cone(f) \to X[1],$$

where $Cone(f) = \mathsf{L}\mathrm{coker}(f)$.

Extra structure:
D comes with a fully faithful embedding A $\to$ D to the degree zero, and the functors H^n : D $\to$ A, $n \in \mathbb{Z}$.
A way to formalize this extra structure is to introduce t*-structures*.

t-structures

Definition

Given a triangulated category C, a *t-structure* on C is a pair of full subcategories $(\mathsf{C}^{\leq 0}, \mathsf{C}^{\geq 0})$ satisfying:

(T1) $\mathsf{C}^{\leq -1} := \mathsf{C}^{\leq 0}[1] \subseteq \mathsf{C}^{\leq 0}, \quad \mathsf{C}^{\geq 1} := \mathsf{C}^{\geq 0}[-1] \subseteq \mathsf{C}^{\geq 0}$

(T2) for all $X \in \mathsf{C}^{\leq 0}$ and all $Y \in \mathsf{C}^{\geq 1}$, $\mathrm{Hom}(X, Y) = 0$

(T3) for all $X \in \mathsf{C}$ there is a d.t.

$$X_{\leq 0} \to X \to X_{\geq 1} \xrightarrow{+1}$$

with $X_{\leq 0} \in \mathsf{C}^{\leq 0}$ and $X_{\geq 1} \in \mathsf{C}^{\geq 1}$.

The *heart* of the t-structure $(\mathsf{C}^{\leq 0}, \mathsf{C}^{\geq 0})$ is

$$\mathsf{C}^{\heartsuit} = \mathsf{C}^{\leq 0} \cap \mathsf{C}^{\geq 0}.$$

Example

Let A be a Grothendieck category, $\mathsf{D} = \mathsf{D}(\mathsf{A})$ or $\mathsf{D}^b(\mathsf{A})$.
Set

$$\mathsf{D}^{\leq 0} = \{X \in \mathsf{D} \mid H^n(X) = 0 \text{ for all } n > 0\},$$

$$\mathsf{D}^{\geq 0} = \{X \in \mathsf{D} \mid H^n(X) = 0 \text{ for all } n < 0\}.$$

This gives a t-structure with $\mathsf{D}^{\heartsuit} \approx \mathsf{A}$.

Truncations:

$$\begin{tikzcd}
\dots X^{-2} \arrow[r] \arrow[d, equal] & X^{-1} \arrow[r, "(\partial^{-1})'"] \arrow[d, equal] & \operatorname{Ker}\partial^0 \arrow[d, hook] & & & \\
\dots X^{-2} \arrow[r] & X^{-1} \arrow[r, "\partial^{-1}"] & X^0 \arrow[r, "\partial^0"] & X^1 \arrow[r, "\partial^1"] \arrow[d, two heads] & X^2 \arrow[r] \arrow[d, equal] & X^3 \dots \arrow[d, equal] \\
& & & X^1/\operatorname{Im}\partial^0 \arrow[r, "\overline{\partial^1}"] & X^2 \arrow[r] & X^3 \dots
\end{tikzcd}$$

Example

Let A be an Abelian category. A *torsion pair* in A is a pair of full subcategories (T, F) such that

(T1) for all $X \in \mathsf{T}$ and all $Y \in \mathsf{F}$, $\mathrm{Hom}(X, Y) = 0$

(T2) for all $X \in \mathsf{C}$ there is a s.e.s. $0 \to T \to X \to F \to 0$ with $T \in \mathsf{T}$ and $F \in \mathsf{F}$.

For example, for $\mathsf{A} = \mathsf{Ab}$,
$\mathsf{T} =$ torsion groups, $\mathsf{F} =$ torsion-free groups, ,
$\mathsf{T} = n^\infty$-torsion groups, $\mathsf{F} =$ groups with torsion coprime to n
Then there is a t-structure on $\mathsf{D} = \mathsf{D}(\mathsf{A})$ or $\mathsf{D}^b(\mathsf{A}))$ given by

$$\mathsf{D}^{\leq 0} = \{X \in \mathsf{D} \mid H^1(X) \in \mathsf{T}, H^j(X) = 0 \text{ for } j \geq 2\},$$

$$\mathsf{D}^{\geq 0} = \{X \in \mathsf{D} \mid H^0(X) \in \mathsf{F}, H^j(X) = 0 \text{ for } j \leq -1\}.$$

Few facts:

- One has $\mathsf{C}^{\leq 0} = \{X \in \mathsf{C} \mid \mathsf{C}(X,Y) = 0 \quad \forall Y \in \mathsf{C}^{\geq 1}\}$
  and $\mathsf{C}^{\geq 0} = \{X \in \mathsf{C} \mid \mathsf{C}(X,Y) = 0 \quad \forall Y \in \mathsf{C}^{\leq -1}\}$
- Any (co-)represented functor $\mathsf{C}(-,A), \mathsf{C}(B,-)$ is *cohomological*: Any d.t. $X \to Y \to Z \overset{+1}{\to}$ induces a l.e.s.

  $$\cdots \to \mathsf{C}(Z,A) \to \mathsf{C}(Y,A) \to \mathsf{C}(X,A) \to \mathsf{C}(Z[-1],A) \to \ldots$$

- Consequently, $\mathsf{C}^{\leq 0}, \mathsf{C}^{\geq 0}$ are both "closed under extensions": Given a d.t. $X \to Y \to Z \overset{+1}{\to}$ with $X, Z \in \mathsf{C}^{\leq 0}$ and any $A \in \mathsf{C}^{\geq 1}$, the induced exact sequence

  $$0 = \mathsf{C}(Z,A) \to \mathsf{C}(Y,A) \to \mathsf{C}(X,A) = 0$$

  shows that $Y \in \mathsf{C}^{\leq 0}$.

Few facts:

- The assignments $X \mapsto X_{\leq 0},\ X \mapsto X_{\geq 1}$ in the d.t. $X_{\leq 0} \to X \to X_{\geq 1} \overset{+1}{\to}$ are functorial, and the resulting functor

$$\tau^{\leq 0} : \mathsf{C} \to \mathsf{C}^{\leq 0} \quad (\tau^{\geq 1} : \mathsf{C} \to \mathsf{C}^{\geq 1}, \text{ resp.})$$

  is a right (left, resp.) adjoint to $\mathsf{C}^{\leq 0} \subseteq \mathsf{C}$ ($\mathsf{C}^{\geq 1} \subseteq \mathsf{C}$, resp.): for $A \in \mathsf{C}^{\leq 0}$ we have the exact sequence

$$0 = \mathsf{C}(A, X_{\geq 1}[-1]) \to \mathsf{C}(A, X_{\leq 0}) \overset{\sim}{\to} \mathsf{C}(A, X) \to \mathsf{C}(A, X_{\geq 1}) = 0$$

- Using shifts, one can define, for all $n \in \mathbb{Z}$,

$$\tau^{\leq n} : \mathsf{C} \to \mathsf{C}^{\leq n}, \quad \tau^{\geq n} : \mathsf{C} \to \mathsf{C}^{\geq n}.$$

- One has $\tau^{\geq n, \leq m} = \tau^{\geq n} \circ \tau^{\leq m} = \tau^{\leq m} \circ \tau^{\geq n} : \mathsf{C} \to \mathsf{C}^{\leq n} \cap \mathsf{C}^{\geq m}$
- When $n = m = 0$ this becomes ${}^{t}H^{0} : \mathsf{C} \to \mathsf{C}^{\heartsuit}$.

Theorem

Let $(\mathsf{C}^{\leq 0}, \mathsf{C}^{\geq 0})$ be a t-structure on C.

1. *The heart $\mathsf{C}^{\heartsuit}$ is an abelian category.*
2. *$0 \to X \to Y \to Z \to 0$ is a ses in $\mathsf{C}^{\heartsuit}$ if and only if $X \to Y \to Z \xrightarrow{+1}$ is a d.t.*

Sketch of proof:

1) *biproducts:* $X, Y \in \mathsf{C}^{\heartsuit} \Rightarrow X \oplus Y \in \mathsf{C}^{\heartsuit} \Rightarrow \mathsf{C}^{\heartsuit}$ is additive.

Sketch of proof:

2) *kernels, cokernels:* Let $X, Y \in \mathsf{C}^\heartsuit$ and $f : X \to Y$.
d.t. $X \xrightarrow{f} Y \to Z \xrightarrow{+1} \quad \rightsquigarrow \quad \underbrace{Y}_{\in \mathsf{C}^\heartsuit} \to Z \to \underbrace{X[1]}_{\in \mathsf{C}^{\leq 0} \cap \mathsf{C}^{\geq -1}} \xrightarrow{+1}$

$\Rightarrow Z \in \mathsf{C}^{\leq 0} \cap \mathsf{C}^{\geq -1}$, hence ${}^tH^0(Z) = \tau^{\geq 0} Z$ and ${}^tH^0(Z[-1]) = \tau^{\geq 0} Z[-1]$.

For $A \in \mathsf{C}^\heartsuit$, applying $\mathsf{C}(-, A)$ yields

$$\begin{tikzcd}
\mathsf{C}(X[1], A) \arrow[r] \arrow[d, equal] & \mathsf{C}(Z, A) \arrow[r] \arrow[d, equal] & \mathsf{C}(Y, A) \arrow[r] \arrow[d, equal] & \mathsf{C}(X, A) \arrow[d, equal] \\
0 \arrow[r] & \mathsf{C}(\tau^{\geq 0} Z, A) \arrow[r] & \mathsf{C}(Y, A) \arrow[r, "- \circ f"] & \mathsf{C}(X, A)
\end{tikzcd}$$

$\Rightarrow \tau^{\geq 0} Z = {}^tH^0(Z) = \operatorname{coker} f$; similarly, ${}^tH^0(Z[-1]) = \ker f$.

3) (*image=coimage:* skipped)

How to produce t-structures on D:

1. Shifting: If $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ is a t-structure, then so is
$$(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})[n] = (\mathsf{D}^{\leq -n}, \mathsf{D}^{\geq -n})$$
for any $n \in \mathbb{Z}$.
2. Tilting: If $t = (\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ is a t-structure and (T, F) is a torsion pair on $\mathsf{D}^{\heartsuit}$, then there is a t-structure $t' = ((\mathsf{D}^{\leq 0})', (\mathsf{D}^{\geq 0})')$ on D given by
$$(\mathsf{D}^{\leq 0})' = \{X \in \mathsf{D} \mid {}^tH^1(X) \in \mathsf{T}, {}^tH^j(X) = 0 \text{ for } j \geq 2\},$$
$$(\mathsf{D}^{\geq 0})' = \{X \in \mathsf{D} \mid {}^tH^0(X) \in \mathsf{F}, {}^tH^j(X) = 0 \text{ for } j \leq -1\}.$$
3. Gluing.

Theorem

Consider adjunctions by triangulated functors

$$\begin{tikzcd}
\mathsf{D}_Z \arrow[r, "i_*" description] & \mathsf{D} \arrow[l, shift right=4ex, "i^*"'] \arrow[l, shift left=4ex, "i^!"] \arrow[r, "j^*" description] & \mathsf{D}_U \arrow[l, shift right=4ex, "j_!"'] \arrow[l, shift left=4ex, "j_*"]
\end{tikzcd}$$

such that the adjunction maps assemble into d.t.'s

$$j_!j^*X \to X \to i_*i^*X \xrightarrow{+1},$$
$$i_*i^!X \to X \to j_*j^*X \xrightarrow{+1}.$$

Let $(\mathsf{D}_Z^{\leq 0}, \mathsf{D}_Z^{\geq 0})$, $(\mathsf{D}_U^{\leq 0}, \mathsf{D}_U^{\geq 0})$ *be* t*-structures on* D_Z *and* D_U*. Then*

$$\mathsf{D}^{\leq 0} = \{X \in \mathsf{D} \mid i^*X \in \mathsf{D}_Z^{\leq 0}, j^*X \in \mathsf{D}_U^{\leq 0}\},$$
$$\mathsf{D}^{\geq 0} = \{X \in \mathsf{D} \mid i^!X \in \mathsf{D}_Z^{\geq 0}, j^*X \in \mathsf{D}_U^{\geq 0}\}$$

is a t*-structure on* D.

Stratifications, constructible derived category

Let X be a complex smooth algebraic variety, $F \supseteq \mathbb{Q}$ a field. A *stratification* Λ of X is given by a chain of closed subvarieties

$$\Lambda : \qquad X = X_0 \supseteq X_1 \supseteq X_2 \cdots \supseteq X_n.$$

We say that Λ' *refines* Λ if Λ' contains all the terms of Λ. Strata of Λ are the connected components of $X_i \setminus X_{i+1}$ and are assumed to be smooth. Define

$$\mathsf{D}^b_{c,\Lambda}(X) := \mathsf{D}^b_{c,\Lambda}(X^{an}, F) = \{X \in \mathsf{D}^b(X^{an}, F) \mid H^i(X) \text{ are locally constant sheaves on the strata of } \Lambda\}.$$

Clearly $\mathsf{D}^b_{c,\Lambda}(X) = \mathsf{D}^b_{loc}(X)$ when $\Lambda = \{X\}$.

Then:

1. When Λ' refines Λ, there is an inclusion $\mathsf{D}^b_{c,\Lambda}(X) \subseteq \mathsf{D}^b_{c,\Lambda'}(X)$.
2. Every pair of stratifications Λ, Λ' allows for a common refinement Λ'', and
$$\mathsf{D}^b_c(X) = \varinjlim_{\Lambda} \mathsf{D}^b_{c,\Lambda}(X).$$
3. Given stratification(s)
$$\Lambda : \quad X \supseteq \underbrace{X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n}_{\Sigma},$$
one has the gluing situation
$$\begin{tikzcd} \mathsf{D}^b_{c,\Sigma}(X_1) \arrow[r, "i_*"] & \mathsf{D}^b_{c,\Lambda}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathsf{D}^b_{loc}(X \setminus X_1) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}.$$

Recall *Verdier duality:*
Let $p : X \to \{*\}$ be the projection onto a point, where X is smooth. Let

$$\omega_X = \mathsf{R}p^! \underline{F}_{\{*\}} = \underline{F}_X[2\dim X].$$

Then the Verdier duality functor $\mathbb{D}$ is given as

$$\mathbb{D} = \mathsf{R}\underline{Hom}(-, \omega_X) : D^b(X) \to D^b(X).$$

- $\mathsf{D}^b_c(X)$ is preserved under $\mathbb{D}$.
- $\mathsf{D}^b_{c,\Lambda}(X)$ is preserved under $\mathbb{D}$ when Λ is so-called *Whitney stratification*. Every stratification can be refined to a Whitney stratification.
- When $\mathcal{L}$ is a local system, $\mathbb{D}(\mathcal{L}) \simeq \mathcal{L}^\vee[2\dim X]$. In particular, If $Loc(X)$ is the category of local systems, then $\mathbb{D}(Loc(X)[\mathrm{dimX}]) = Loc(X)[\mathrm{dimX}]$.

Construction

Let Λ be a stratification of X.

1. For a stratum $U_i := X_i \setminus X_{i+1}$, of dimension d_i, set

$$\mathsf{D}_i^{\leq 0} = \{X \in \mathsf{D}^b_{loc}(U_i) \mid \mathcal{H}^j(X) = 0 \text{ for } j > -d_i\},$$
$$\mathsf{D}_i^{\geq 0} = \{X \in \mathsf{D}^b_{loc}(U_i) \mid \mathcal{H}^j(X) = 0 \text{ for } j < -d_i\},$$

so that $(\mathsf{D}_i^{\leq 0}, \mathsf{D}_i^{\geq 0}) =$
$$\left(\left(\mathsf{D}^b(U_i^{an}, F)^{\leq 0}, \mathsf{D}^b(U_i^{an}, F)^{\geq 0}\right)_{std} \cap \mathsf{D}^b_{loc}(U_i)\right)[d_i]$$
is a t-structure on $\mathsf{D}^b_{loc}(U_i)$ with heart $\approx Loc(U_i)[d_i]$.

Construction

Let Λ be a stratification of X.

2. Inductively on the stratification Λ, $(\mathsf{D}_i^{\leq 0}, \mathsf{D}_i^{\geq 0})$ glue up to a t-structure $(\mathsf{D}_\Lambda^{\leq 0}, \mathsf{D}_\Lambda^{\geq 0})$ on $\mathsf{D}_{c,\Lambda}(X)$, given also as

$$\mathsf{D}_\Lambda^{\leq 0} = \{X \in \mathsf{D}_{c,\Lambda}^b(X) \mid \iota_i^* X \in \mathsf{D}_i^{\leq 0} \text{ for all } i\},$$
$$\mathsf{D}_\Lambda^{\geq 0} = \{X \in \mathsf{D}_{c,\Lambda}^b(X) \mid \iota_i^! X \in \mathsf{D}_i^{\geq 0} \text{ for all } i\},$$

   where $\iota_i : U_i \to X$ is the inclusion.

3. The t-structures $(\mathsf{D}_\Lambda^{\leq 0}, \mathsf{D}_\Lambda^{\geq 0})$ are compatible under refinement. Setting

$${}^p\mathsf{D}^{\leq 0} := \varinjlim_{\Lambda} \mathsf{D}_\Lambda^{\leq 0}, \quad {}^p\mathsf{D}^{\geq 0} := \varinjlim_{\Lambda} \mathsf{D}_\Lambda^{\geq 0},$$

   $({}^p\mathsf{D}^{\leq 0}, {}^p\mathsf{D}^{\geq 0})$ is a t-structure on $\mathsf{D}_c^b(X)$, called the *perverse t-structure*.

Definition

The category of perverse sheaves $Perv(X)$ is defined as ${}^p\mathrm{D}^{\heartsuit}$, the heart of $({}^p\mathrm{D}^{\leq 0}, {}^p\mathrm{D}^{\geq 0})$.

- $Perv(X)$ is Abelian.
- The fact that $\mathbb{D}(Loc(X)[\dim X]) = Loc(X)[\dim X]$ translates to the fact that $Perv(X)$ is stable under Verdier duality.
- semiperversity = being member of ${}^p\mathrm{D}^{\leq 0}$, Verdier dual being semiperverse = being member of ${}^p\mathrm{D}^{\geq 0}$.