PLANAR GRAPHS

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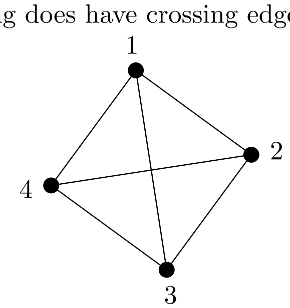
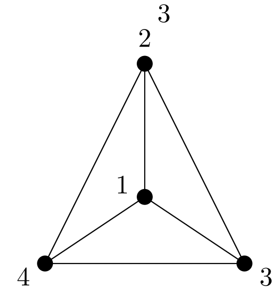
(This material is a supplement to Sections 10.7-10.8 of Rosen's book.)

1. Planar graphs

A curve is a subset of the plane of the form $\{(x, y) \mid x = f(t), y = g(t), 0 \le t \le 1\}$, where f and g are continuous functions. A graph is *planar* if it *can* be drawn in the plane so that edges are represented by curves which don't cross (except at vertices). For example, we can see that the complete graph K_4 is planar using second drawing, even though the first drawing does have crossing edges.





However, we will see that there is no way to draw K_5 without crossings, so it isn't planar.

Given a planar graph, the plane is divided into disjoint regions bounded by the edges. Let r be the number of these, including the outside. For example, in the second drawing of K_4, we have 3 triangles 123, 124, 134 plus the outside, so $r = 4$. The cycle C_n is planar, with one inner region and one outer, so $r = 2$.

THEOREM 1.1 (Euler)**.** *If G is a connected planar graph with v vertices, e edges and r regions, then*

$$v - e + r = 2$$

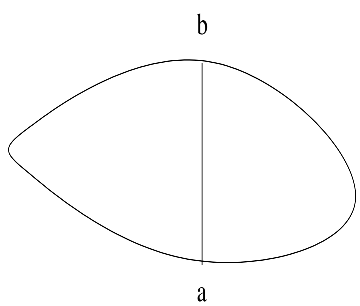
or equivalently

$$r = 2 - v + e$$

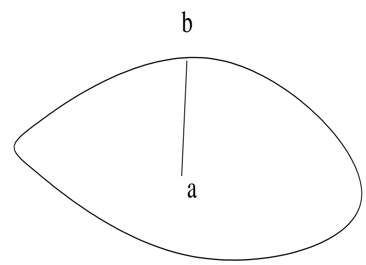
I should probably point out that terms like "drawing" or "region" haven't been defined precisely. The proof given below will also use some intuitive arguments in a couple of places. However, a completely rigorous treatment would require a branch of math called *topology*, which I don't want to get into.

Proof. We prove this by induction on e. The base case $e = 0$ would mean G consists of a single vertex, otherwise it wouldn't be connected. Clearly $r = 1$ so $v - e + r = 1 - 0 + 1 = 2$.

Now assume that the theorem holds for any connected graph with fewer than e vertices. Choose an edge ϵ of G, with endpoints a, b, such that G remains connected after removing it. Let G' denote the new graph obtained from G by removing ϵ, and possibly an end point a or b, if it becomes isolated, because we want G' to be connected. We can always relabel things so that a is the vertex removed. We refer to this as the second case. The first case is where no vertices are removed.



Case 1



Case 2

Let v', e', r' denote the number of vertices, edges and regions for G'. We have $e' = e - 1$. So by induction, we know that $v' - e' + r' = 2$. We have $v = v'$ in the first case. In this case, e will subdivide a region of G'. So removing e drops number of regions to $r' = r - 1$. So we obtain $v - (e - 1) - (r - 1) = 2$, which proves the formula in the first case. In the second case, $v' = v - 1$. In this case, e does not subdivide a region of G', so $r' = r$. Therefore $v - 1 - (e - 1) - r = 2$, which proves what we want in the second case. □

COROLLARY 1.2. *A tree (a connected graph with no simple circuits) satisfies* $v = e + 1$.

Proof. There is one outer region and no interior regions, so $r = 1$. The result now follows from Euler's theorem. □

Given a planar graph G, the dual graph G^*, which is really a multigraph, has vertices corresponding to regions of G. Two vertices of G^* are connected by n edges in G^* if the regions have n edges in G as a common boundary. Also a vertex in G^* has a loop for each bridge of G contained within the corresponding region. See figure 1. Recall that a bridge of G is an edge for which G would become disconnected after deleting it. The degree of a region is the degree of the corresponding vertex of G^*. In simple cases, when there no bridges in R, $\deg R$ is the number of regions adjacent to R, or equivalently the number of edges around the perimeter of R. The number of vertices and edges of G^* are $v^* = r$ and $e^* = e$. The handshaking theorem applied to G^* tells us that

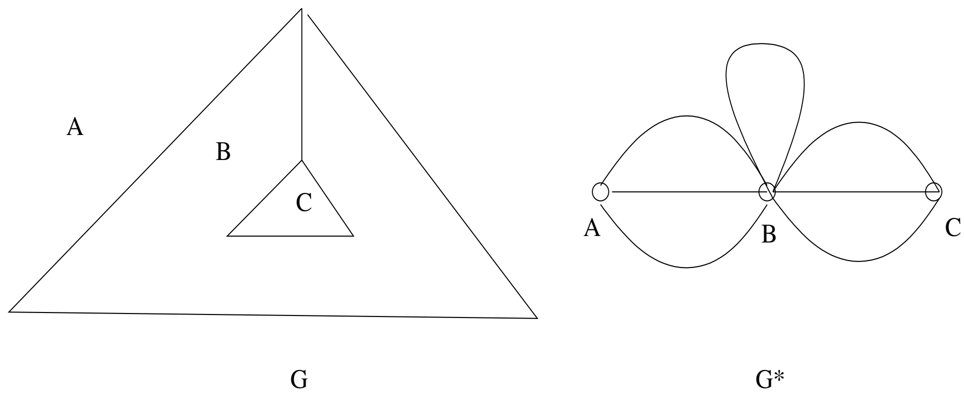


FIGURE 1. Dual graph

THEOREM 1.3 (Handshaking theorem, version 2)**.**

$$\sum_{regions} \deg R = 2e$$

EXAMPLE 1.4. *One can check that this holds for the graph in figure 1. The degrees of A, B, C are 3, 8 and 3. For B, we have to remember that the loop contributes 2 to its degree. These add up to $2e = 2(7)$*

As a corollary to Euler's theorem, we have

THEOREM 1.5. *If G is a connected simple planar graph with at least 3 vertices*

$$e \leq 3v - 6$$

Proof. Note that since each interior region R is at least a triangle, $\deg R \geq 3$. This also holds for the outer region because of the assumption about the number of vertices (although this isn't completely obvious). Therefore

$$2e = \sum \deg R \geq 3r$$

or

$$r \leq \frac{2}{3}e$$

So that

$$2 - v + e \leq \frac{2}{3}e$$

Therefore

$$\frac{1}{3}e \leq v - 2$$

Now multiply by 3 to get the desired result. □

This can be used to show that certain graphs are not planar.

EXAMPLE 1.6. *K_5 is not planar because $v = 5$ and $e = \binom{5}{2} = 10$ is greater than $3v - 6$.*

$K_{3,3}$ is also not planar, but the above test fails in the sense that the above inequality holds. We can tweak the result to handle this.

THEOREM 1.7. *Suppose that G is a connected simple planar graph. Assume that all cycles have length at least g and*

$$v \geq \frac{g}{2} + 1$$

then

$$e \leq \frac{g(v-2)}{g-2}$$

When $g = 3$, we get

$$e \leq 3v - 6$$

when $v \geq 3$ as above.

Proof. If G is a tree, then $e = v - 1$.

$$v - 1 \leq \frac{g(v-2)}{g-2}$$

is equivalent to

$$(g-2)(v-1) \leq g(v-2)$$

After rearranging terms, we get

$$2g - (g-2) \leq gv - (g-2)v$$

A bit more algebra reduces this to

$$v \geq \frac{g}{2} + 1$$

If G is not a tree, then any region R is borders a circuit. This means $\deg R \geq g$ by assumption. Therefore

$$2e = \sum \deg R \geq gr$$

or

$$r \leq \frac{2}{g} e$$

From Euler, we get

$$2 - v + e \leq \frac{2}{g} e$$

which simplifies to

$$e \leq \frac{g}{g-2}(v-2)$$

□

COROLLARY 1.8. *If G has no cycles of length 3 and $v \geq 3$, then*

$$e \leq 2v - 4$$

Proof. We can take $g = 4$ in the theorem.

□

EXAMPLE 1.9. *We can now show that $K_{3,3}$ is not planar. Since it has no triangles, so we can apply the theorem with $g = 4$. We see that $v = 6, e = 9$ breaks the inequality.*

We can see that any graph which contains K_5 or $K_{3,3}$ is not planar. A theorem of Kuratowski shows that this essentially the only way that a graph can fail to be planar. The precise statement, but not the proof, can be found in Rosen.

2. Regular polyhedra

Instead drawing the graph on the plane, we could draw it on the sphere. Using stereographic projection, explained in class, we can see such a graph would be planar. One way to get a graph on sphere is to take a polyhedron which is a sort of 3D polygon. The cube is the most familiar example. Some examples are pictured below. The quantity r would be the number of faces or polygon sides in the picture.

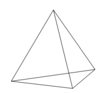


Figure 2. tetrahedron

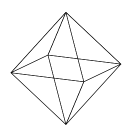


Figure 3. octahedron

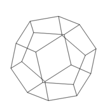


Figure 4. dodecahedron

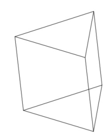


FIGURE 5. prism

The first three examples exhibit a high degree of symmetry, and the prism has less. A polyhedron is called *regular* if the degree of each vertex is the same, and all the faces have the same number of sides. The last condition means that the degree of each region is the same. So it can be viewed as a property of planar graphs. The first three examples are regular, but the prism isn't. A classical theorem, which was known to Euclid, is:

THEOREM 2.1. *There are only five regular polyhedra up to isomorphism: the tetrahedron (4 triangular faces), the cube (6 square faces), the octahedron (8 triangular faces), the dodecahedron (12 pentagonal faces), or the icosahedron (20 triangular faces).*

Proof. We will prove weaker statement, that the type and number of faces is as indicated. We are assuming that the degrees of vertices are a constant n and the degrees of regions is another constant m. Both of these integers are at least 3. The handshaking theorems imply

$$nv = 2e = mr$$

or

$$v = \frac{2}{n}e, \quad r = \frac{2}{m}e$$

Plugging into Euler and simplifying gives

$$e(\frac{1}{n} + \frac{1}{m} - \frac{1}{2}) = 1$$

We must have

$$\frac{1}{n} + \frac{1}{m} - \frac{1}{2} > 0$$

There are few possibilities. For instance if $m, n \geq 4$, it will fail. So one of them, say $m = 3$, then

$$\frac{1}{n} + \frac{1}{3} - \frac{1}{2} = \frac{1}{n} - \frac{1}{6} > 0$$

so $n = 3, 4, 5$. In this way, we can see that there only five solutions $(m, n) = (3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$. In the last case, when $m = 5, n = 3$

$$e(\frac{1}{5} + \frac{1}{3} - \frac{1}{2}) = 1$$

forces

$$e = 30, v = \frac{2}{3}e = 20, r = \frac{2}{5}e = 12$$

This means there are 12 pentagons. This is the dodecahedron. One can work out the other numbers to see that $(3,3)$ corresponds to a tetrahedron, $(3,4)$ to a cube, $(3,5)$ to a icosahedron, and $(4,3)$ to an octahedron.

□

3. Five color theorem.

Given a map, what is the minimal number of colors need so that adjacent countries have different colors?
Answer: 4

We can translate this into a graph theory problem: given a graph G, the chromatic number $\chi(G)$ is the minimum number of colors so that adjacent vertices have different colors. The following theorem was conjectured in the 1800s and the eventual solution involved a reduction to a finite, but large, number of cases checked by computer in the 1970s.

THEOREM 3.1 (Appel-Haken)**.** *A planar graph has chromatic number at most 4.*

We prove an easier version.

THEOREM 3.2. *A planar graph has chromatic number at most 5.*

Proof. We prove it by induction on the number of vertices. Suppose that G be the planar graph. We claim that there is a vertex with degree at most 5. Suppose not. Then all vertices have degree at least 6. Using the hand shaking theorem

$$e = \frac{1}{2}\sum \deg(x) \geq 3v$$

But this contradicts the previous inequality $e \leq 3v - 6$ for planar graphs. This proves the claim that there exists a vertex x with $\deg x \leq 5$. Let's assume that the degree is 5 for simplicity. Label the adjacent vertices $x_1, \ldots, x_5$ in order as you go around x. By induction, we can color $G - x$ with five colors labelled $1, \ldots, 5$. Let's assume that x_i is colored with color i. Let G_{13} be the subgraph colored with colors 1 and 3. If x_1, x_3 lie on different connected components of $G_{1,3}$. Then we can change color 3 to 1 without affecting anything. Now color x with 3, and we are done. We can argue with x_2, x_4 the same way. If they lie in different components of $G_{2,4}$, we can solve the problem as before.

There is one remaining case that x_1, x_3 lie on the same component of $G_{1,3}$, and x_2, x_4 will lie on the same component of $G_{2,4}$. Then we can find a path P connecting x_1 and x_3 lying in $G_{1,3}$, and a path P' in $G_{2,4}$ connecting x_2 and x_4. The paths P and P' will have to cross somewhere (see figure 6 below). But this is impossible, because points on P and P' must have different colors; 1 or 3 for P and 2 or 4 for P'.

□

In general, the chromatic number is hard to compute, but here a few easy cases:

EXAMPLE 3.3. *We can see that $\chi(K_n) = n$ because any two vertices are adjacent, so they need different colors. Since K_4 is planar, this shows that we can't do better than 4 colors.*

EXAMPLE 3.4. *If G is bipartite then $\chi(G) = 2$ essentially by definition.*

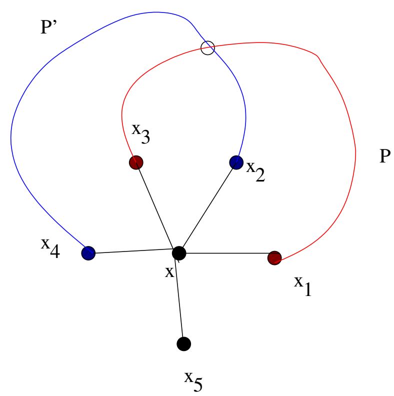


FIGURE 6. 5 coloring theorem

EXAMPLE 3.5. *If n is even, then $\chi(C_n) = 2$ because we can alternate colors.*