LINEAR RECURRENCES AND MATRICES

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This supplements the discussion in section 8.2 of Rosen. I will assume that you have taken/are taking/will take linear algebra (MA265 or MA351 or equivalent).

1. Linear Recurrences

Recall that a homogeneous linear recurrence of order k is a sequence $a_0, a_1, a_2 \ldots$ satisfying

$$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} \tag{1}$$

for $n \geq k$. The first k values $a_0, \ldots, a_{k-1}$ are called initial values. We will refer to the rest of the sequence as a solution. There a couple of easy observations, which we state as propositions.

PROPOSITION 1.1. *Once the initial values are specified, the rest of the sequence, i.e. the solution, is uniquely determined.*

The first statement should be obvious. The second follows from linearity:

PROPOSITION 1.2. *If a_n and a'_n are two solutions to (1), then $da_n + ea'_n$ is also a solution for any choice of numbers d and e.*

2. Linear recurrences using matrices

Recall that a matrix is a rectangular arrangment of numbers. The only ones we need are 2×2 matrices, such as the first matrix below, 2×1 matrices such as the second. The product of two such matrices is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

We can also multiply two 2×2 matrices by

$$AB = (A(\text{1st col of } B) \quad A(\text{2nd col of } B))$$

Let's consider a second order recurrence

$$a_{n+2} = c_1 a_{n+1} + c_2 a_n, \quad n \geq 0 \tag{2}$$

with initial values a_0, a_1. Introduce matrices

$$v_n = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$$

$$C = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix}$$

Then

$$Cv_n = \begin{pmatrix} c_1 a_{n+1} + c_2 a_n \\ a_{n+1} \end{pmatrix}$$

Therefore (2) is equivalent to the matrix equation

$$v_{n+1} = Cv_n$$

So that

$$v_1 = Cv_0, v_2 = Cv_1 = C^2 v_0, \ldots$$

We immediate get the complete solution

THEOREM 2.1. *The solution to the recurrence* (2) *is given by*

$$v_n = C^n v_0$$

This solution is both good and bad. Good because it's conceptually very simple, and bad because $C^n v_0$ is generally hard to calculate. There is one case, where it is easy. If v_0 happens to be an *eigenvector* for C, then

$$Cv_0 = \lambda v_0$$

for some number λ called the *eigenvalue*. In this case we can see that

COROLLARY 2.2. *If v_0 is an eigenvector with eigenvalue λ, then*

$$v_n = \lambda^n v_0$$

In general, we shouldn't expect v_0 to be an eigenvector. However, what is generally true is that v_0 is a linear combination $v_0 = b_1 v' + b_2 v''$ of eigenvectors v', v'' with eigenvalues λ_1 and λ_2 respectively. In this case, using proposition 1.2, we see that:

COROLLARY 2.3. *If $v_0 = b_1 v' + b_2 v''$ as above, then*

$$v_n = \lambda_1^n v' + \lambda_2^n v''$$

While the eigenvectors and eigenvectors can be found in principle, it seems easier to find an elementary solution from scratch, using the above solution as a guide to what to look for.

3. Elementary solution

The first step to finding a solution to (2) is to first look for a solution of the form

$$a_n = \lambda^n$$

When $n = 2$, (2) says that

$$\lambda^2 = c_1 \lambda + c_2 \lambda$$

or that

$$\lambda^2 - c_1 \lambda - c_2 = 0 \tag{3}$$

This is called the *characteristic equation* for the recurrence. It is not hard to see that it happens to also be the characteristic equation for the matrix C, if you remember what that means.

THEOREM 3.1. *If λ is a solution to* (3)*, then $a_n = \lambda^n$ is a solution to* (2)*.*

Proof. Multiplying (3) by λ^n yields

$$\lambda^{n+2} - c_1 \lambda^{n+1} - c_2 \lambda^n = 0$$

But this implies (2). □

This is a quadratic equation, so it can be solved using highschool algebra. Either we get two distinct solutions λ_1, λ_2 or just one solution. Let us assume the first case. Then

THEOREM 3.2. *If (3) has two distinct solutions λ_1, λ_2, then all the solutions to (2) are given by*

$$a_n = b_1\lambda_1^n + b_2\lambda_2^n$$

for constants b_1, b_2.

Proof. It follows from proposition 1.2 and theorem 3.1 that

$$a_n = b_1\lambda_1^n + b_2\lambda_2^n$$

is a solution to (2). So we just have to show that any solution is of this form. Suppose that a_0, a_1 are given initial values. Then we have to solve the system of equations

$$b_1 + b_2 = a_0$$
$$\lambda_1 b_1 + \lambda_2 b_2 = a_1$$

where the b_i are the unknowns. This can be done using Cramer's rule from linear algebra, which gives the solution as a ratio of determinants. The only thing to check is the determinant in the denominator

$$\det\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \lambda_2 - \lambda_1$$

is nonzero. But this follows from the fact that $\lambda_1 \neq \lambda_2$. □

Examples and other things were done in class.